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An L^2 theory for differential forms on path spaces I ☆

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Abstract

An L^2 theory of differential forms is proposed for the Banach manifold of continuous paths on a Riemannian manifold M furnished with its Brownian motion measure. Differentiation must be restricted to certain Hilbert space directions, the H-tangent vectors. To obtain a closed exterior differential operator the relevant spaces of differential forms, the H-forms, are perturbed by the curvature of M. A Hodge decomposition is given for L^2 H-one-forms, and the structure of H-two-forms is described. The dual operator d^* is analysed in terms of a natural connection on the H-tangent spaces. Malliavin calculus is a basic tool.

Keywords: Path space; L^2 cohomology; Hodge decomposition; Malliavin calculus; Banach manifolds; Bismut tangent spaces; Markovian connection; Itô map; Infinite dimensional; Curvature; Exterior products; Differential forms

1. Introduction

Background. We are concerned with the construction of an L^2 Hodge theory on path spaces with respect to a suitable reference measure and a collection of 'admissible' vector fields. Consider the space of continuous paths on a compact Riemannian manifold, over a fixed time interval $[0, T]$. Path spaces are Banach manifolds with the usual concepts of differentiable functions and

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differential forms, for example see Eells [24], Eliasson [25], Lang [54]. They also have a natural measure, their *Brownian motion*, or *Wiener* measure.

From the works of Bismut [10], Léandre [47], Driver [20] and others following pioneering work by L. Gross [44] in the classical Wiener space case, it seems the natural Sobolev differential calculus for functions on path spaces using such measures is of differentiation in directions given by Hilbert spaces of tangent vectors at each point: essentially the tangent vectors of finite energy. These are the so-called Bismut tangent spaces. The integration by parts formula given by Driver [20], and subsequent results suggest that these notions will lead to a satisfactory, and useful, Malliavin type calculus in this context. However the construction of differential form theory using Bismut tangent spaces leads to difficulties even at the level of the definition of exterior derivative. This is because of the lack of integrability of Bismut tangent 'bundle': the Lie bracket of suitable Bismut tangent space valued vector fields does determine a vector field, but in the presence of curvature it no longer takes values in the Bismut tangent spaces. Several ways of getting round this problem have been formulated, and carried out, especially by Léandre [55,56, 58] who gave analytical de Rham groups and showed that they agree with the singular cohomology of the spaces. See also [57]. But we are not aware of any which have led to an L^2 theory with Hodge–Kodaira Laplacian on our path spaces in the presence of curvature. In flat Wiener space the problem does not arise and the L^2 theory was defined and shown to be cohomologically trivial by Shigekawa [69,70]. See also Mitoma [62] and Arai and Mitoma [5]. For abstract Wiener manifolds, a class of infinite dimensional manifolds with an integrable Hilbert bundle of admissible directions, see Piech [65]. For M a compact Lie group with bi-invariant metric the corresponding results were proved by Fang and Franchi [42], but using the Bismut tangent spaces obtained from the flat left invariant connection on M so the problem again is avoided. They also considered loop groups [42]. For work done on 'sub-manifolds' of Wiener space see Airault and van Biesen [4], van Biesen [71] and especially Kusuoka [52,53], Kazumi and Shigekawa [48]. These submanifolds were constructed to replicate loop spaces over Riemannian manifolds, with their natural "Brownian bridge" measures. For a general survey see Léandre [59], and for a more introductory article concentrating on the approach taken here, see [34].

Let M be a compact C^∞ Riemannian manifold. For a fixed positive number T, consider the space $C_{x_0}M$ of continuous paths $\sigma : [0, T] \to M$ starting at a given point x_0 of M, furnished with its natural structure as a C^∞ Banach manifold and Brownian motion measure μ_{x_0}. For smooth differential forms there are the de Rham cohomology groups $H^q_{de\ Rham}(C_{x_0}M)$. C.J. Atkin informs us that the techniques of [7,8] can be extended to show that the de Rham groups would be equal to the singular cohomology groups, even though $C_{x_0}M$ does not admit smooth partitions of unity, and so trivial for $q \geqslant 0$ since based path spaces are contractible. For related work, also see Lempert and Zhang [60] on Dolbeault cohomology of a loop space. Since our primary interest is in the differential analysis associated with the Brownian motion measure μ on $C_{x_0}M$, which could equally well be considered on Hölder paths of any exponent smaller than a half, we could use Hölder rather than continuous paths and it is really only for notational convenience that we do not. In that case we would have smooth partitions of unity, see Bonic, Frampton and Tromba [11]. However contractibility need not imply triviality of the de Rham cohomology groups when some restriction is put on the spaces of forms. For example if $f : \mathbf{R} \to \mathbf{R}$ is given by $f(x) = x$ then df determines a non-trivial class in the first bounded de Rham group of $\mathbf{R}$. If f has value $+1$ for $x > 1$ and -1 for $x < 1$ then df is non-trivial in L^2-cohomology. In finite dimensions the L^2-cohomology of a cover $\tilde{M}$ of a compact manifold M gives important topological invariants of M even when $\tilde{M}$ is contractible, e.g. see Atiyah [6]; note also Bueler and Prokhorenkov [12], Ahmed and Stroock [1], and Gong and Wang [43].

The Bismut tangent spaces H^1_σ are defined by the parallel translation

$$/\!\!/_t(\sigma): T_{x_0}M \to T_{\sigma(t)}M$$

of the Levi-Civita connection and consist of those $v \in T_\sigma C_{x_0}M$ such that $v_t = /\!\!/_t(\sigma)h_t$ for $h_\cdot \in L_0^{2,1}([0,T]; T_{x_0}M)$. To have a satisfying L^2 theory of differential forms on $C_{x_0}M$ the obvious choice would be to consider 'H-forms,' i.e. for 1-forms these would be ϕ with $\phi_\sigma \in (H^1_\sigma)^*$, $\sigma \in C_{x_0}M$, and this agrees with the natural H-derivative $d_{\mathcal{H}}f$ for $f: C_{x_0}M \to \mathbf{R}$. For L^2 q-forms the obvious choice would be ϕ with $\phi_\sigma \in \bigwedge^q (H^1_\sigma)^*$, using here the Hilbert space completion for the exterior product. An L^2 de Rham theory would come from the complex of spaces of L^2 sections

$$\cdots \xrightarrow{\bar d} L^2\Gamma \bigwedge\nolimits^q \big(H^1_\sigma\big)^* \xrightarrow{\bar d} L^2\Gamma \bigwedge\nolimits^{q+1} \big(H^1_\sigma\big)^* \xrightarrow{\bar d} \cdots \tag{1.1}$$

where $\bar d$ would be a closed operator obtained by closure from the usual exterior derivative: for V^j, $j = 1$ to $q+1$, C^1 vector fields, and ϕ a differentiable one-form:

$$\begin{aligned} d\phi\big(V^1 \wedge \cdots \wedge V^{q+1}\big) &= \frac{1}{q+1}\sum_{i=1}^{q+1}(-1)^{i+1} L_{V^i}\big[\phi\big(V^1 \wedge \cdots \wedge \widehat{V^i} \wedge \cdots \wedge V^{q+1}\big)\big] \\ &\quad + \frac{1}{q+1}\sum_{1 \leqslant i<j \leqslant q+1} (-1)^{i+j}\phi\big(\big[V^i, V^j\big] \wedge V^1 \wedge \cdots \wedge \widehat{V^i} \wedge \cdots \wedge \widehat{V^j} \wedge \cdots \wedge V^{q+1}\big) \end{aligned} \tag{1.2}$$

where $[V^i, V^j]$ is the Lie bracket and $\widehat{V^j}$ means omission of the vector field V^j.

From this would come the de Rham–Hodge–Kodaira Laplacians $\bar d \bar d^* + \bar d^* \bar d$ and an associated Hodge decomposition. However the brackets $[V^i, V^j]$ of sections of $H^1_\cdot$ are not in general sections of $H^1_\cdot$, e.g. see Cruzeiro and Malliavin [18], Driver [21], see also [33], and formula (1.2), below, for d does not make sense for ϕ_σ defined only on $\bigwedge^q H^1_\sigma$, each σ, as mentioned earlier.

Our proposal is to replace the Hilbert spaces $\bigwedge^q H^1_\sigma$ in (1.1) by a family of different Hilbert spaces $\mathcal{H}^q_\sigma$, $q = 2, 3, \ldots$, continuously included in $\bigwedge^q T_\sigma C_{x_0}M$, though keeping the exterior derivative a closure of the classical exterior derivative on smooth cylindrical forms.

In Elworthy and Li [32], for $q = 1, 2$, we identified a class of Hilbert subspaces $\mathcal{H}^q_\sigma$, of the completed exterior powers $\bigwedge^q T_\sigma C_{x_0}$ of the tangent space $T_\sigma C_{x_0}$ to $C_{x_0}M$ at a path σ which could be the basic building blocks of an L^2 de Rham and Hodge theory for $C_{x_0}M$. We described $\mathcal{H}^2_\sigma$ without proof, proved closability of exterior differentiation on corresponding L^2 1-forms, defined a self-adjoint Hodge–Kodaira Laplacian on such L^2 1-forms and established the Hodge decomposition.

The article [33] both discusses some of the constructions here for more general diffusion measures and connections, and relates them to the Bismut type formulae for differential forms on M [31], see also Driver and Thalmaier [22]. In particular it shows that a very natural class of two-vector fields on $C_{x_0}M$ are of the type we consider here (i.e. are sections of $\mathcal{H}^2$).

Main results. Here we give a detailed analysis of $\mathcal{H}^2_\sigma$ and define $\mathcal{H}^q_\sigma$ for $q > 0$. For $q = 1$, as a space $\mathcal{H}^1_\sigma = H^1_\sigma$. For flat manifolds, $\mathcal{H}^q_\sigma = \bigwedge^q H^1_\sigma$ for all q and the standard Hodge decomposition theorem follows. However in general, the spaces $\mathcal{H}^q_\sigma$ we construct are different from

$\bigwedge^q H$, the exterior products of the Bismut tangent bundle. Sections of $\mathcal{H}^q$ are called H-q-vector fields and sections of $(\mathcal{H}^q)^*$ are called H-differential forms of degree q. In fact $\mathcal{H}^2_\sigma$ is a deformation of $\bigwedge^2 H^1_\sigma$ inside $\mathbf{L}_{skew}(\mathcal{H}^1_\sigma, \mathcal{H}^1_\sigma)$ by the curvature of M. As a Hilbert space $\mathcal{H}^2_\sigma$ is defined to be isometric to $\bigwedge^2 H^1_\sigma$ by a map involving the curvature of the so-called damped Markovian connection on the Bismut tangent "bundle." Algebraic operations such as interior products acting on H-two-vectors, and the exterior products of H-one-forms, as well as the derivation property for the exterior derivative are shown to make sense. A Hodge decomposition is given for H-one-forms. In a sequel, Part II, we establish the analogous decomposition for L^2 2-forms, and we show that the spaces $\mathcal{H}^q_\sigma$ defined by suitable Itô maps $\mathcal{I}$ depend only on the Riemannian structure of the base manifold M.

Organisation. The article is organised as follows:

Section 2. Review of basic results concerning exterior powers of relevant spaces of tangent vectors to $C_{x_0}M$.
Section 3. Special Itô maps and the definition of $\mathcal{H}^q$.
Section 4. Characterisation of $\mathcal{H}^1$ and $\mathcal{H}^2$.
Section 5. H-one-forms: exterior differentiation and Hodge decomposition.
Section 6. Tensor products as operators: algebraic operations on H-one-forms.
Section 7. The derivation property of $\bar{d}^1$.
Section 8. Infinitesimal rotations as divergences.
Section 9. Differential geometry of the space $\mathcal{H}^2$ of two-vectors.
Appendix A. Conventions.
Appendix B. Brackets of vector fields, torsion, and $d\phi(v^1 \wedge v^2)$.

In Section 2 we discuss the various completed tensor products of tangent, and other spaces which we will use. Properties of these relating to tensor products of abstract Wiener spaces are used in order to define our spaces $\mathcal{H}^q$ in Section 3. The aim is to show that these constructions are well behaved and have interesting geometry.

One of the main results, see Section 4, is a characterisation of $\mathcal{H}^2$ as a perturbation of $\bigwedge^2 \mathcal{H}$ by a curvature of the Levi-Civita connection on M. Write $\mathcal{H}_\sigma = \mathcal{H}^1_\sigma$, then

$$\mathcal{H}^2_\sigma = (I + Q_\sigma)\bigwedge^2 \mathcal{H}_\sigma \tag{1.3}$$

for some operator Q_σ on $\bigwedge^2_\epsilon T_\sigma C_{x_0}$. Equivalently

$$u \in \mathcal{H}^2 \quad \text{if and only if} \quad u - \mathbb{R}(u) \in \bigwedge^2 \mathcal{H}$$

where $\mathbb{R}$ is identified in Section 9 as the curvature of the damped Markovian connection on the H-tangent spaces.

In Section 5 we rapidly recall the results concerning closability of our exterior derivative on H-one-forms and the Hodge decomposition for H-one-forms.

The remainder, the main part, of the article is an analysis of the space $\mathcal{H}^2$, its associated H-two-forms, and the adjoint of the exterior derivative, an operator from H-two-forms to H-one-forms, together with the corresponding divergence operator from two-vector fields to vector fields. In Section 6 it is shown that the exterior product of two H-one-forms is naturally an H-two-form, and the interior product of an H-two-form with a H-one-form is a H-one-form. The operator Q has image in $\mathcal{L}_{skew}(\mathcal{H};\mathcal{H})$, which implies an element of $\mathcal{H}^2_\sigma$ can be considered to be an element of $\mathcal{L}_{skew}(\mathcal{H}_\sigma;\mathcal{H}_\sigma)$, cf. Corollary 6.2, although in general it is not compact and so not in $\bigwedge^2\mathcal{H}_\sigma$. In Section 7 a corresponding derivation formula for the exterior derivative of H-one-forms, Theorem 7.1, is shown to hold.

In Section 8 it is shown that the elements of the image of suitable smooth sections of $\bigwedge^2\mathcal{H}$ by Q "have a divergence" in the sense of satisfying an integration by parts formula and a formula is given in 8.1 for div $Q(V^1\wedge V^2)$. Vector fields which are not H-vector fields also make their appearance, especially as Lie brackets. The latter involve infinitesimal rotations which "have a divergence," and in their case the divergence is zero. It is natural to ask if they themselves are divergences, in this extended sense, of some two-vector field. In Section 8 this is shown to be true in a wide class of adapted situations on flat Wiener space, Proposition 8.2. This has independent interest, but it is extended, in Theorem 9.3, to show that the torsion of the damped Markovian connection when applied to suitable non-anticipating H-vector fields is the divergence of the perturbing factor in the definition of $\mathcal{H}^2$:

$$\operatorname{div} Q\big(u^1\wedge u^2\big)=\frac{1}{2}\mathbb{T}\big(u^1,u^2\big). \tag{1.4}$$

Here $\mathbb{T}$ is the torsion of the damped Markovian connection $\breve{\nabla}$. This helps explain the "cancellation" of the bracket occurring with our exterior derivative, and fits in with the result of Cruzeiro and Fang [16], concerning the vanishing of the divergence of such torsions. The damped Markovian connection, introduced by Cruzeiro and Fang [16], plays an important role here, as it did in [35]. As in [35] we introduce it by giving a $C_{id}([0,T];O(n))$-bundle structure to $\mathcal{H}$. This is done in Section 9. Here we also relate the divergence of our H-two-vector fields to the adjoint of the damped Markovian covariant derivative in a non-anticipating situation, Corollary 9.7: For suitable non-anticipating U, V,

$$\breve{\nabla}^*(U\wedge V)=\operatorname{div}(I+Q)(U\wedge V). \tag{1.5}$$

We also describe the curvature of the damped Markovian connection in Section 9D, to establish our claim that $\mathcal{H}^2$ is a perturbation of $\bigwedge^2\mathcal{H}^1$ using this curvature operator, Theorem 4.3(iii). In Section 9D we essentially show that $\mathbb{D}^{2,1}$ H-two-forms are in the domain of the adjoint of $\bar{d}^{1*}$, extending the result for one-forms proved in [35].

List of symbols

$C_{x_0}M$ or C_{x_0}—space of continuous paths over M starting from x_0.
$T_\sigma C_{x_0}M$ or $T_\sigma C_{x_0}$—tangent space at σ to $C_{x_0}M$.
$\mathcal{H}^1_\sigma$ or $\mathcal{H}_\sigma$—Bismut tangent space, a Hilbert space included in $T_\sigma C_{x_0}$.
$\mathcal{H}^1$ or $\mathcal{H}$—corresponding Bismut tangent "bundle," $\bigcup\mathcal{H}^1_\sigma$.
$\mathcal{H}^2$—vector "bundle" with fibres $\mathcal{H}^2_\sigma\subset\bigwedge^2 T_\sigma C_{x_0}M$.
ΓB—sections of a vector bundle B.
$L^2\Gamma B$—L^2 sections of a vector bundle B.
$C_0\mathbf{R}^m$—Wiener space with Wiener measure $\mathbf{P}$, the canonical probability space.
$L^{2,1}_0(G)$—for G a Hilbert space, this is $\{h:[0,T]\to G$ such that $\int_0^T|\dot{h}_s|^2\,ds<\infty\}$. When $G=\mathbf{R}^m$, this is the Cameron–Martin space, denoted by H.
$(\xi_t, t\geqslant 0)$—a Brownian stochastic flow of diffeomorphisms of M.

$T\xi_t$—space derivative of ξ_t.
μ—Brownian motion measure, also called Wiener measure, on $C_{x_0}M$.
$\mathcal{I}$—the Itô map induced by $(\xi_t(x_0),\ t \geqslant 0)$, $\mathcal{I}(\omega) := \xi_\cdot(x_0, \omega)$.
$T\mathcal{I}$—H-derivative of the Itô map.
$\mathcal{F}^{x_0}$—the algebra generated by $(\xi_t(x_0), t \geqslant 0)$ on M.
$\overline{f}(\sigma)$—conditional expectation of f given $\mathcal{I} = \sigma$, $\sigma \in C_{x_0}$, e.g. $\overline{T\mathcal{I}}_\sigma$.
$W_t^{(q)}$—Weitzenbock flow of q-vectors, Eq. (4.1).
${W^{(q)}}_t^s$—Weitzenbock flow starting from time s.
W_t—damped parallel translation, $W_t = W_t^{(1)}$.
$\frac{\mathbb{D}^q}{dt}$, $\frac{\mathbb{D}}{dt}$—see Section 4.
$L_2T_\sigma C_{x_0}$—the space of L^2 tangent vectors at σ, Definition 4.1.
$\mathcal{W}$—isometry between $\mathcal{H}^1$ and $L_2TC_{x_0}$, Eq. (4.3).
$\mathcal{L}(E_1; E_2)$—the space of continuous linear maps between Banach spaces.
$\mathcal{L}_2(H_1; H_2)$—Hilbert–Schmidt maps between Hilbert spaces.
$\mathcal{R}$, $\mathcal{R}^q$, Ric—respectively the curvature operator, the Weitzenbock curvature on q forms, and the Ricci curvature on M.

In general we shall use $|\ |$ to denote norms of finite dimensional spaces. $\|\ \|$ for infinite dimensional spaces, with $\boldsymbol{|\ |}$ for spaces such as $L^2(\Omega; \mathbf{R}^n)$, or $L^2(C_{x_0}M; \mathbf{R})$, where integration over probability spaces are involved.

2. Exterior powers: Notation

For convenience the conventions we use for tensor products, exterior powers, etc. are gathered together as Appendix A. Please note that they differ from those used in our previous articles, such as [32].

A. All linear spaces are over $\mathbf{R}$. We shall deal with tensor products of Hilbert spaces and of Banach spaces of continuous paths. For any linear space E let $\bigotimes_0^q E$ denote the qth algebraic tensor product of E with itself and $\bigwedge_0^q E$ the linear subspace of antisymmetric elements. For infinite dimensional Banach spaces E we will need completions of these spaces, e.g. see Ruston [67] or Cigler, Losert and Michor [14]:

(i) When $E = T_\sigma C_{x_0}$ or $C_0\mathbf{R}^m$ let $\bigotimes^q E$ and $\bigwedge^q E$ denote the completions using the largest cross norm, i.e. the projective tensor products $\|-\|_\pi$. For general Banach spaces E_i, if v is in the algebraic tensor product $E_1 \otimes_0 \cdots \otimes_0 E_q$,

$$\|v\|_\pi = \inf\left\{\sum_{i=1}^n \prod_{k=1}^q \|a_i^k\|,\ \text{where } v = \sum_{i=1}^n \bigotimes_{k=1}^q a_i^k,\ a_i^k \in E_k,\ n < \infty\right\}.$$

(ii) When E is a Hilbert space H, let $\bigotimes^q H$ and $\bigwedge^q H$ denote the standard Hilbert space completions, (so $\bigotimes^2 H$ can be identified with the space of Hilbert–Schmidt operators on H).

(iii) In general let $\bigotimes_\varepsilon^q E$ and $\bigwedge_\varepsilon^q E$ refer to the completions with respect to the smallest reasonable cross norm, i.e. the inductive cross norm,

$$\|w\|_\epsilon = \sup_{\|u_k^*\|_{E^*} \leqslant 1,\, u_k^* \in E^*} \left|\left(u_1^* \otimes \cdots \otimes u_q^*\right)(w)\right|.$$

We shall use the natural inclusion maps as identifications and so consider

$$\bigotimes_0^q E \subset \bigotimes^q E \subset \bigotimes_\varepsilon^q E.$$

Thus a differential q-form ϕ on $C_{x_0}M$ which by definition gives a continuous antisymmetric multilinear map $\phi_\sigma : T_\sigma C_{x_0} \times \cdots \times T_\sigma C_{x_0} \to \mathbf{R}$, Lang [54], can equivalently be defined as a section of the bundle $\mathcal{L}(\bigwedge^q TC_{x_0}; \mathbf{R})$ with fibres the dual spaces $(\bigwedge^q T_\sigma C_{x_0})^*$, $\sigma \in C_{x_0}M$.

B. If $S : E_1 \to E_2$ and $T : F_1 \to F_2$ are two linear maps of linear spaces, there is the induced linear map $S \otimes T : E_1 \otimes_0 F_1 \to E_2 \otimes_0 F_2$. The Banach space constructions are functorial so that if $S, T \in \mathcal{L}(C_0\mathbf{R}^m; T_\sigma C_{x_0})$ then $S \otimes T$ determines a continuous linear map of the completed tensor spaces $\bigotimes^2 C_0\mathbf{R}^m$ to $\bigotimes^2 T_\sigma C_{x_0}M$ and if $S = T$ we have its restriction $\bigwedge^2 S : \bigwedge^2 C_0\mathbf{R}^m \to \bigwedge^2 T_\sigma C_{x_0}M$, Ruston [67, p. 63] and Cigler, Losert and Michor [14]; with the corresponding result for the inductive tensor product, for the Hilbert space case, and for $q > 2$. There is also the estimate on operator norms

$$\|S^1 \otimes \cdots \otimes S^q\| \leqslant \|S^1\| \cdot \cdots \cdot \|S^q\|$$

so that in particular

$$\left\|\bigwedge^q S\right\| \leqslant \|S\|^q$$

in all of these cases, see Ruston [67] and Cigler, Losert and Michor [14].

For example let $H \equiv L_0^{2,1}\mathbf{R}^m$ be the (Cameron–Martin) Hilbert space of functions $h : [0, T] \to \mathbf{R}^m$ of the form $h_t = \int_0^t \dot{h}_s \, ds$ with $\dot{h} \in L^2([0, T]; \mathbf{R}^m)$ and inner product $\langle h^1, h^2 \rangle = \int_0^T \langle \dot{h}_s^1, \dot{h}_s^2 \rangle_{\mathbf{R}^m} \, ds$. Thus the indefinite integral

$$\int_0^{\cdot} : L^2\bigl([0, T]; \mathbf{R}^m\bigr) \to H$$

is an isometry with inverse which we will write as

$$\frac{d}{d\cdot} : H \to L^2\bigl([0, T]; \mathbf{R}^m\bigr).$$

From this we obtain the isometry

$$\bigwedge^q \left(\int_0^{\cdot} \right) : \bigwedge^q L^2\bigl([0, T]; \mathbf{R}^m\bigr) \to \bigwedge^q H$$

with inverse

$$\bigwedge^q \left(\frac{d}{d\cdot} \right) : \bigwedge^q L_0^{2,1}\mathbf{R}^m \to \bigwedge^q L^2\bigl([0, T]; \mathbf{R}^m\bigr).$$

C. We will regularly make use of the well-known isometries

$$\bigotimes_{\varepsilon}^{q} C_0\mathbf{R}^m \xrightarrow{\rho} C_0\big([0,T]^q; \bigotimes^q \mathbf{R}^m\big)$$

where the right-hand side consists of those continuous $\alpha : [0,T]^q \to \bigotimes^q \mathbf{R}^m$ for which $\alpha(t_1,\dots,t_q) = 0$ if $t_j = 0$ for any j. For example see Cigler, Losert and Michor [14, p. 66]. For $V \in \bigotimes_{\varepsilon}^{q} C_0\mathbf{R}^m$, write

$$V_{t_1,\dots,t_q} := \rho(V)(t_1,\dots,t_q).$$

Let $\mathrm{ev}_t : C_0\mathbf{R}^m \to \mathbf{R}^m$ be the evaluation map at time t, then

$$V_{t_1,\dots,t_q} = (\mathrm{ev}_{t_1} \otimes \cdots \otimes \mathrm{ev}_{t_q})V.$$

Also note that such V lies in $\bigwedge_{\varepsilon}^{q} C_0\mathbf{R}^m$ if and only if $\rho(V) : [0,T]^q \to \bigotimes^q \mathbf{R}^m$ anti-commutes with permutations, i.e.

$$V_{t_{\pi(1)},\dots,t_{\pi(q)}} = (-1)^{\pi} S_{\pi} V_{t_1,\dots,t_q}$$

for any permutation π on $\{1,\dots,q\}$ with S_π the induced action on $\bigotimes^q \mathbf{R}^m$. If so,

$$V_{t,\dots,t} \in \bigwedge^q \mathbf{R}^m$$

and

$$V_{t_1,t_2,\dots,t_2} \in \mathbf{R}^m \otimes \bigwedge^{q-1} \mathbf{R}^m,$$

etc. From this we see that elements of $\bigwedge_{\varepsilon}^{q} C_0\mathbf{R}^m$ and hence those of the smaller spaces $\bigwedge^q C_0\mathbf{R}^m$ are determined by their values on the simplex $0 \leqslant t_1 \leqslant \cdots \leqslant t_q \leqslant T$.

Similarly, to any $V \in \bigotimes_{\varepsilon}^{q} T_\sigma C_{x_0}$ we have $V_{t_1,\dots,t_q} \in T_{\sigma_{t_1}} M \otimes \cdots \otimes T_{\sigma_{t_q}} M$ corresponding to an isometric isomorphism of $\bigotimes_{\varepsilon}^{q} T_\sigma C_{x_0}$ with the space of continuous maps $V_{\cdot}$ on $[0,T]^q$ such that

$$\begin{tikzcd} & \bigotimes^q TM \arrow[d, "\pi"] \\ {[0,T]^q} \arrow[ur, "V"] \arrow[r, "\sigma_\cdot \times \cdots \times \sigma_\cdot"'] & M \times \cdots \times M \end{tikzcd}$$

commutes and $V_{t_1,\dots,t_q} = 0$ when $t_j = 0$ for any j.

D. By functorality the inclusion $i : L_0^{2,1}\mathbf{R}^m \to C_0\mathbf{R}^m$ gives rise to a continuous linear inclusion $\bigotimes^q i : \bigotimes^q H \to \bigotimes_{\varepsilon}^{q} C_0\mathbf{R}^m$. From paragraph **B** we see that $V \in \mathrm{Image} \bigotimes^q i$ if and only if

$$V_{t_1,\dots,t_q} = \int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_q} U_{s_1,\dots,s_q}\, ds_1 \dots ds_q, \tag{2.1}$$

$(t_1,\dots,t_q)\in[0,T]^q$, for some $U_\cdot\in L^2([0,T]^q;\bigotimes\mathbf{R}^m)$. Here we use the isometry ρ of $\bigotimes^q L^2([0,T];\mathbf{R}^m)$ with $L^2([0,T]^q;\bigotimes^q\mathbf{R}^m)$. In fact the $U_\cdot$ in the above formula is just $\rho(\bigotimes^q(\frac{d}{d\cdot})V)$ or equivalently $U_{t_1,\dots,t_q}$ is the weak derivative $\frac{\partial^q}{\partial t_1\dots\partial t_q}V_{t_1,\dots,t_q}$.

E. Given a bounded linear operator $S:E\to F$ of Banach spaces there is also the functorial construction

$$\big(d\bigotimes{}^q\big)(S):\bigotimes{}_0^q E\to\bigotimes{}_0^q F$$

defined by linearity and

$$\begin{aligned}&\big(\big(d\bigotimes{}^q\big)(S)\big)\big(e^1\otimes\cdots\otimes e^q\big)\\&\quad=S\big(e^1\big)\otimes e^2\otimes\cdots\otimes e^q+e^1\otimes S\big(e^2\big)\otimes\cdots\otimes e^q+\cdots+e^1\otimes e^2\otimes\cdots\otimes S\big(e^q\big).\end{aligned}$$

This is just a sum of operators described in paragraph **B** and so extends over the relevant completion. The same notation will be kept for these extensions.

Note that if V is in $\bigotimes^q H$ then $((d\bigotimes^q)(\frac{d}{d\cdot}))(V)$ is in $\bigotimes^q L^2([0,T];\mathbf{R}^m)$ with kernel

$$\left(\big(d\bigotimes{}^q\big)\left(\frac{d}{d\cdot}\right)\right)(V)_{t_1,\dots,t_q}=\sum_{j=1}^q\frac{\partial}{\partial t_j}V_{t_1,\dots,t_q}. \tag{2.2}$$

The restriction $(d\Lambda^q(S))$ of $(d\bigotimes^q(S))$ to $\bigwedge_0^q E$ has the form

$$\big(d\Lambda^q(S)\big)\big(v^1\wedge v^2\wedge\cdots\wedge v^q\big)=S\big(v^1\big)\wedge v^2\wedge\cdots\wedge v^q+\cdots+v^1\wedge v^2\wedge\cdots\wedge S\big(v^q\big)$$

and for $q=2$

$$\big(d\Lambda^2(S)\big)\big(v^1\wedge v^2\big)=\frac{1}{2}\big\{Sv^1\otimes v^2+v^1\otimes Sv^2-Sv^2\otimes v^1-v^2\otimes Sv^1\big\}. \tag{2.3}$$

3. Special Itô maps and the definition of $\mathcal{H}_\sigma^q$

A. Take a surjective C^∞ vector bundle morphism, $X:\underline{\mathbf{R}}^m\to TM$, of the trivial $\mathbf{R}^m$ bundle over M onto TM, for some $m\geqslant n=\dim M$. Suppose that X induces the given Riemannian metric on M and let Y be the $\mathbf{R}^m$-valued 1-form such that $Y_x=X(x)^*:T_xM\to\mathbf{R}^m$. For U a vector field and $v\in T_xM$, set

$$\nabla_vU=X(x)\,d\big[y\to Y_yU(y)\big](v), \tag{3.1}$$

as in Elworthy, LeJan and Li [29,30], where it was called LW connection for X. Suppose that the connection ∇ is the Levi-Civita connection. Take (B_t) to be the canonical Brownian motion on $\mathbf{R}^m$ with probability space $C_0\mathbf{R}^m$ and Wiener measure $\mathbb{P}$ and consider the stochastic differential equation on M

$$dx_t=X(x_t)\circ dB_t,\quad 0\leqslant t\leqslant T. \tag{3.2}$$

Then the solutions are Brownian motions on M. Let μ_{x_0} be the Brownian motion measure on $C_{x_0}M$, the probability distribution of the solution starting from x_0. An example is the gradient system induced from an isometric immersion $\alpha : M \to \mathbf{R}^m$ with $X(x) : \mathbf{R}^m \to T_x M$ defined to be the orthogonal projection for each $x \in M$. Another class of examples arises from symmetric space structures on M, see [30].

For our fixed x_0 in M there is the solution map, or *Itô map*,

$$\mathcal{I} : C_0\mathbf{R}^m \to C_{x_0}M,$$

of (3.2) defined by

$$\mathcal{I}(\omega)_t = x_t(\omega), \quad \omega \in C_0\mathbf{R}^m,$$

where x_t is the solution starting at x_0. Thus $\mathcal{I}_*(\mathbb{P}) = \mu_{x_0}$. This Itô map has an H-derivative in the sense of Malliavin calculus which is a continuous linear map from the Cameron–Martin space $H \equiv L_0^{2,1}\mathbf{R}^m$,

$$T_\omega\mathcal{I} : H \to T_{\mathcal{I}(\omega)}C_{x_0},$$

for almost all $\omega \in C_0\mathbf{R}^m$. Thus for $h \in H$ and $0 \leqslant t \leqslant T$ we have $T\mathcal{I}(h)_t \in T_{x_t}M$, a.s.

B. Let $\{\xi_t\colon\ 0 \leqslant t \leqslant T\}$ denote the flow of (3.2) so $\mathcal{I}(\omega)_t = x_t(\omega) = \xi_t(x_0, \omega)$. It can be taken to consist of random C^∞ diffeomorphisms $\xi_t : M \to M$ with derivative maps $T\xi_t : TM \to TM$, so that $T_{x_0}\xi_t \in \mathcal{L}(T_{x_0}M; T_{x_t}M)$.

Take $h \in H$. Set $v_t = T\mathcal{I}(h)_t$. Bismut showed that $v_\cdot$ satisfies the covariant equation along the paths of $\{x_t\colon\ 0 \leqslant t \leqslant T\}$

$$Dv_t = \nabla_{v_t}X \circ dB_t + X(x_t)\dot{h}_t\,dt \tag{3.3}$$

with solution

$$v_t = T\xi_t \int_0^t (T\xi_s)^{-1}\big(X(x_s)\dot{h}_s\big)\,ds. \tag{3.4}$$

Lemma 3.1. *(See [30,38].) There is a canonical decomposition of the noise $\{B_t\colon\ 0 \leqslant t \leqslant T\}$ given by*

$$dB_t = \tilde{/\!\!/}_t\,d\tilde{B}_t + \tilde{/\!\!/}_t\,d\beta_t \tag{3.5}$$

where

(i) $\{\tilde{B}_t\colon\ 0 \leqslant t \leqslant T\}$ *is a Brownian motion on the orthogonal complement* $[\ker X(x_0)]^\perp$ *of the kernel of* $X(x_0)$ *in* $\mathbf{R}^m$;

(ii) $\{\beta_t\colon\ 0 \leqslant t \leqslant T\}$ *is a Brownian motion on* $\ker X(x_0)$;

(iii) *for each* $t \geqslant 0$, $\tilde{/\!\!/}_t : C_{x_0}M \to O(m)$ *is a measurable map into the orthogonal group of* $\mathbf{R}^m$ *with* $\tilde{/\!\!/}_t(\sigma)[\ker X(x_0)] = \ker X(\sigma_t)$ *for* μ_{x_0} *almost all* $\sigma \in C_{x_0}M$.

N.B. We will regularly consider random variables on $C_{x_0}M$, such as $\tilde{/\!/}_t$, to be random variables on $C_0\mathbf{R}^m$ by taking their composition with $\mathcal{I}$. For example the stochastic equation (3.5) above is to be interpreted that way. Moreover let $\mathcal{F}^{x_0}$ be the σ-algebra on $C_0\mathbf{R}^m$ generated by $\mathcal{I}$ with $\{\mathcal{F}_t^{x_0},\ 0 \leqslant t \leqslant T\}$ the filtration generated by $(x_s\colon 0 \leqslant s \leqslant T)$. Then we can, and often will, consider $\mathcal{F}^{x_0}$-measurable functions as functions, defined up to equivalence, on $C_{x_0}M$.

Let $\mathcal{F}^\beta$ be the σ-algebra generated by $\{\beta_t\colon 0 \leqslant t \leqslant T\}$, and $\mathcal{F}^{\tilde{B}}$ that generated by $\{\tilde{B}_t\colon 0 \leqslant t \leqslant T\}$. From Elworthy and Yor [38], Elworthy, LeJan and Li [30] we know that

(a) $\mathcal{F}^\beta$ and $\mathcal{F}^{\tilde{B}}$ are independent and
(b) $\mathcal{F}^{\tilde{B}} = \mathcal{F}^{x_0}$;
(c) Eq. (3.3) can be written as the Itô equation

$$Dv_t = \nabla_{v_t} X(\tilde{/\!/}_t\, d\beta_t) - \frac{1}{2}\operatorname{Ric}^\#(v_t)\, dt + X(x_t)\dot{h}_t\, dt \tag{3.6}$$

where $\operatorname{Ric}_x^\#\colon T_xM \to T_xM$ corresponds to the Ricci curvature by $\langle \operatorname{Ric}_x^\#(u^1), u^2\rangle = \operatorname{Ric}(u^1, u^2)$ for u^1, u^2 in T_xM.

We shall often write covariant derivatives such as $\nabla_v X$ as $\nabla X(v)$ so $\nabla X(v) \circ dB_t$ is just $\nabla_v X \circ dB_t$.

C. We first show that $\bigwedge^q T_\omega\mathcal{I}$ take values in the exterior product space $\bigwedge^q TC_{x_0}$ rather than just in $\bigwedge_\epsilon^q TC_{x_0}$. Recall that a continuous linear map of H to a separable Banach space E is γ-*radonifying* if it maps the canonical Gaussian cylinder set measure of H to a Borel measure on E. The 2-*summing norm*, $\pi_2(A)$, of an operator $A\colon E \to F$ is given by

$$\pi_2(A)^2 = \sup_{\{x_n\}\subset E} \frac{\sum \|Ax_n\|^2}{\sup_{\|u\|=1,\, u\in E^*} \sum (u(x_n))^2}$$

where $\{x_n\}$ is a finite subset of E. When E and F are Hilbert spaces A has finite two summing norm if and only if A is Hilbert–Schmidt. See for example Pietsch [66].

Lemma 3.2. *For almost all $\omega \in C_0\mathbf{R}^m$ the map*

$$T_\omega\mathcal{I}\colon H \to T_{\mathcal{I}(\omega)}C_{x_0}$$

is γ-radonifying. Its operator norm $\|T\mathcal{I}\|$ is in $L^p(C_0\mathbf{R}^m)$ for $1 \leqslant p < \infty$ as is the 2-summing norm of its adjoint.

Proof. Note that $\alpha\colon h \mapsto \int_0^\cdot (T\xi_s)^{-1} X(x_s)(\dot{h}_s)\, ds$ maps H to $L_0^{2,1}(T_{x_0}M)$ and is continuous linear; almost surely. The inclusion $i\colon L_0^{2,1}(T_{x_0}M) \to C_0T_{x_0}M$ is γ-radonifying. Write $T\mathcal{I} = T\xi_\cdot \circ i \circ \alpha$. Then the first result follows by composition properties of γ-radonifying maps and continuity of $T_{x_0}\xi_\cdot\colon C_0T_{x_0}M \to T_{x_\cdot(\omega)}C_{x_0}$. The pth power integrability of the operator norms come from the corresponding properties of $T\xi_t$ and $(T\xi_t)^{-1}$, e.g. see Kifer [49]. For the 2-summing norm apply Schwartz's duality theorem [68] to see that the adjoint of the γ-radonifying map i

is 2-summing with norm $\pi_2(i)$. Then use the composition properties of 2-summing operators to estimate the 2-summing norm

$$\pi_2\big((T\mathcal{I})^*\big) \leqslant \|\alpha^*\|\pi_2\big(i^*\big)\big\|(T\xi_t)^*\big\|, \quad \text{a.s.}$$

Then apply the integrability results again to see the norm is in L^p. □

Theorem 3.3. *For almost all ω the map $\bigwedge^q T_\omega I$ can be considered as a continuous linear map from the Hilbert space completion of the qth exterior power of H to the projective exterior power of the tangent space $(T_{\mathcal{I}(\omega)}C_{x_0})$*

$$\bigwedge^q(T_\omega\mathcal{I}) : \bigwedge^q\big(L_0^{2,1}\mathbf{R}^m\big) \to \bigwedge^q(T_{\mathcal{I}(\omega)}C_{x_0}).$$

Moreover the operator norms lie in $L^p(C_0\mathbf{R}^m)$ for $1 \leqslant p < \infty$.

Proof. This follows from Lemma 3.2 and results of Carmona and Chevet [13] especially their Proposition 3.1 and Lemma 3.1 a version of which is stated below as Lemma 3.4. Although they only deal with tensor products of two maps the lemma shows that the result holds for general q by induction. □

Denote by $E \otimes_\pi F$ the completion of the tensor product space of two Banach spaces E and F using projective tensor product norm, cf. notation (i) in Section 2A.

Lemma 3.4 *(Carmona and Chevet). Consider separable Hilbert spaces H and K and separable Banach spaces E and F. Let $T : H \to E$ be γ-radonifying and $S : K \to F$ bounded linear. Then $S \otimes T : H \otimes K \to E \otimes_\pi F$ is a bounded linear map into the projective tensor product. Moreover*

$$\|S \otimes T\|_{\mathbb{L}(H\otimes K; E\otimes_\pi F)} \leqslant \pi_2\big(T^*\big)\|S\|$$

where $\pi_2(T^*)$ denotes the 2-summing norm of the adjoint of T.

The conditional expectations of these operators can be defined as in Elworthy and Yor [38], Elworthy, LeJan and Li [30], to give bounded linear maps, defined almost surely,

$$\overline{\bigwedge^q(T\mathcal{I})}(\omega) : \bigwedge^q H \to \bigwedge^q(T_{\mathcal{I}(\omega)}C_{x_0}).$$

For example

$$\overline{\bigwedge^q(T\mathcal{I})}(\omega) := \mathbf{E}\big\{\bigwedge^q(T_\omega\mathcal{I})\big|\mathcal{F}^{x_0}\big\}(\omega)$$

is given by

$$\overline{\bigwedge^q(T\mathcal{I})}(\omega)(h)_t = \big(\bigwedge^q /\!\!/_t\big)\mathbf{E}\big\{\bigwedge^q\big(/\!\!/_t^{-1}\big)\bigwedge^q\big(T\mathcal{I}_t(h)\big)\big|\mathcal{F}^{x_0}\big\}(\omega).$$

For μ_{x_0} almost all $\sigma \in C_{x_0}M$ we have also

$$\big(\overline{\bigwedge^q(T\mathcal{I})}\big)_\sigma : \bigwedge^q H \to \bigwedge^q(T_\sigma C_{x_0})$$

given by

$$\big(\overline{\bigwedge^q(T\mathcal{I})}\big)_\sigma(h) := \mathbf{E}\big\{\bigwedge^q(T.\mathcal{I})(h)\big|\mathcal{I}=\sigma\big\}.$$

Note the inequalities

$$\begin{aligned}\big\|\overline{\bigwedge^q(T\mathcal{I})}(\omega)(h)\big\| &\leqslant \mathbf{E}\big\{\big\|\bigwedge^q(T.\mathcal{I})(h)\big\|\big|\mathcal{F}^{x_0}\big\}(\omega) \quad \text{a.s.}\\ &\leqslant \mathbf{E}\big\|\bigwedge^q(T_\omega\mathcal{I})\big\|\|h\| \quad \text{a.s.}\end{aligned}$$

which give L^p bounds for operator norms of $\overline{\bigwedge^q(T\mathcal{I})}$, $q=1,2,\ldots.$

D. *Definition of $\mathcal{H}^q_\sigma$, H-q-vector fields and H-q-forms.* We can now define $\mathcal{H}^q_\sigma$, for almost all $\sigma\in C_{x_0}M$, to be the image of $\overline{\bigwedge^q(T\mathcal{I})_\sigma}$ in $\bigwedge^q T_\sigma C_{x_0}$ together with the inner product induced by the linear bijection

$$\overline{\bigwedge^q T\mathcal{I}_\sigma}\big|_{[\ker\overline{\bigwedge^q T\mathcal{I}_\sigma}]^\perp} : \big[\ker\overline{\bigwedge^q(T\mathcal{I})_\sigma}\big]^\perp \to \mathcal{H}^q_\sigma.$$

Thus the $\mathcal{H}^q_\sigma$ are Hilbert spaces with natural continuous linear inclusions ι_σ, say, into the $\bigwedge^q T_\sigma C_{x_0}$.

Denote by $\mathcal{H}^q=\bigcup_\sigma\mathcal{H}^q_\sigma$ the "vector bundle over $C_{x_0}M$" with fibres $\mathcal{H}^q_\sigma$, and $(\mathcal{H}^q)^*$ the corresponding dual "bundle." Set $\mathcal{H}=\mathcal{H}^1$. Since these are only almost sure defined it is not strictly speaking correct to consider them as bundles over $C_{x_0}M$ though some vector bundle structure is given to $\mathcal{H}$ in [35] see also Section 9 below. The space of L^2 sections of $\mathcal{H}^q$ and $\mathcal{H}^{q*}$ are denoted by $L^2\Gamma\mathcal{H}^q$ and $L^2\Gamma\mathcal{H}^{q*}$. Sections of $(\mathcal{H}^q)^*$ or of $(\mathcal{H}^q)$ will be called H-q-forms (or admissible q-forms), or H-q-vector fields, respectively. Note that any q-form on $C_{x_0}M$ restricts to give an H-q-form.

4. Characterization of $\mathcal{H}^1_\cdot$ and $\mathcal{H}^2_\cdot$

A. 'Damped parallel translations' $W^{(q)}_t$ will play an essential role. For a q-vector $v\in\bigwedge^q T_{x_0}M$, define $W^{(q)}_t(V)\in\bigwedge^q T_{x_t}M$ to be the random q-vector satisfying

$$\frac{D}{dt}W^{(q)}_t(V) = -\frac12\mathcal{R}^q\, W^{(q)}_t(V),\quad 0\leqslant t\leqslant T, \tag{4.1}$$

where $\mathcal{R}^q\in\mathrm{Hom}(\bigwedge^q TM;\bigwedge^q TM)$ is the Weitzenbock curvature term defined by $\mathcal{R}=\Delta-\operatorname{trace}\nabla^2$, see e.g. Airault [3], Elworthy [26], Ikeda and Watanabe [46], Elworthy, LeJan and Li [30], Elworthy, Li and Rosenberg [36], Malliavin [61]. Here (4.1) is a covariant equation along the paths of our solution $\{x_t\colon 0\leqslant t\leqslant T\}$ to (3.2).

For $q=1$ write W_t for $W^{(1)}_t$. Then $W_t: T_{x_0}M\to T_{x_t}M$ is the Dohrn–Guerra translation given by

$$\frac{D}{dt}W_t(V) = -\frac12\operatorname{Ric}^\#_{x_t}\big(W_t(V)\big),\quad 0\leqslant t\leqslant T.$$

Write

$$\frac{\mathbb{D}}{dt} = W_t \frac{D}{dt} W_t^{-1}$$

acting on suitably regular vector fields $\{v_t\colon 0 \leqslant t \leqslant T\}$ along the paths of $\{x_t\colon 0 \leqslant t \leqslant T\}$. Then

$$\frac{\mathbb{D}}{dt} = \frac{D}{dt} + \frac{1}{2}\mathrm{Ric}^{\#},$$

cf. Fang, formula (1.3), in Fang [41] and Norris [63].

Definition 4.1. For almost all paths ω, define the L^2 tangent space $L^2T_{\sigma.}C_{x_0}$ to consist of those paths $u : [0,T] \to TM$ over σ with

$$/\!/_{.}^{-1} u. \in L^2\big([0,T]; T_{x_0}M\big)$$

together with its natural Hilbert space structure.

It was shown in [28], see also [30,32] that

$$\overline{T\mathcal{I}}_t(h) = \mathcal{W}_t\big(X(x.)\dot{h}.\big) \tag{4.2}$$

where

$$\mathcal{W}: L^2T_{x.}C_{x_0} \to T_{x.}C_{x_0}$$

is defined by

$$\big(\mathcal{W}(u)\big)_t = W_t \int_0^t (W_r)^{-1}(u_r)\,dr. \tag{4.3}$$

Note that

$$\frac{\mathbb{D}}{dt}\big(\mathcal{W}(u)\big)_t = u_t, \quad u \in L^2T_{x.}C_{x_0}. \tag{4.4}$$

Thus, as shown in [30,32],

$$\mathcal{H}^1_\sigma = \big\{v \in T_\sigma C_{x_0}\colon /\!/_{.}^{-1} v. \in L_0^{2,1}(T_{x_0}M)\big\} \tag{4.5}$$

with inner product

$$\big\langle v^1, v^2\big\rangle_{\mathcal{H}^1} = \int_0^T \left\langle \frac{\mathbb{D}}{ds} v_s^1, \frac{\mathbb{D}}{ds} v_s^2\right\rangle ds \tag{4.6}$$

so that $\frac{\mathbb{D}}{d.} : \mathcal{H}^1_\sigma \to L^2T_\sigma C_{x_0}$ is an isometric isomorphism with inverse $\mathcal{W}$ for almost all $\sigma \in C_{x_0}M$. Thus it agrees as a Hilbertable space with the usual Bismut tangent space, though the inner product is not the one originally used. Using the same notation, by Section 2D we note that a vector u of $\bigwedge^2 T_\sigma C_{x_0}M$ is in $\bigwedge^2 \mathcal{H}_\sigma$ if and only if there exists $\underline{k} \in \bigwedge^2 L^2T_\sigma C_{x_0}M$ so that

$$u_{s,t} = \left(\bigwedge^2 \mathcal{W}\right)_{s,t} \underline{k},$$

or written in full,

$$u_{s,t} = \left(W_s \int_0^s (W_{r_1})^{-1}(-)\,dr_1 \otimes W_t \int_0^t (W_{r_2})^{-1}(-)\,dr_2 \right) \underline{k}_{r_1,r_2}. \tag{4.7}$$

If so $\underline{k}_{s,t} = \frac{\mathbb{D}}{\partial s} \otimes \frac{\mathbb{D}}{\partial t} u$ or equally $\underline{k} = \bigwedge^2 \frac{\mathbb{D}}{d\cdot} u$.

B. More generally let $L^2(\bigwedge^q TM)_\sigma$ and $C_0(\bigwedge^q TM)_\sigma$ denote respectively the spaces of L^2 and continuous paths vanishing at 0, $u : [0,T] \to \bigwedge^q TM$ over σ. Define

$$\mathcal{W}^{(q)} : L^2\left(\bigwedge^q TM\right)_\sigma \to C_0\left(\bigwedge^q TM\right)_\sigma$$

by

$$\left(\mathcal{W}^{(q)}(V_\cdot)\right)_t = W_t^{(q)} \int_0^t \left(W_r^{(q)}\right)^{-1}(V_r)\,dr \tag{4.8}$$

$$= \int_0^t W^{(q)}{}_t^r (V_r)\,dr \tag{4.9}$$

where

$$W^{(q)}{}_t^s = W_t^{(q)} \left(W_s^{(q)}\right)^{-1}$$

is the solution to

$$\frac{D}{dt} W^{(q)}{}_t^s (V) = -\frac{1}{2} \mathcal{R}^q \left(W^{(q)}{}_t^s (V) \right), \quad s \leqslant t \in [0,T], \tag{4.10}$$

with $W^{(q)}{}_s^s = \mathrm{Id} : \bigwedge^q T_{\sigma_s} M \to \bigwedge^q T_{\sigma_s} M$. Write W_t^s for $W^{(1)}{}_t^s$ and observe that $\mathcal{W}^{(1)} = \mathcal{W}$. For simplicity we shall write $\mathcal{W}_t^{(q)}(V_\cdot)$ for $(\mathcal{W}^{(q)}(V_\cdot))_t$.

Set

$$\frac{\mathbb{D}^{(q)}}{dt} = \left(\frac{D}{dt}\right) + \frac{1}{2} \mathcal{R}^q, \tag{4.11}$$

acting on q-vectors on M along a sample path σ. Then as for $q = 1$, and for $W_t^{(q)}$ defined by (4.10):

$$\frac{\mathbb{D}^{(q)}}{dt} V_{t,\dots,t} = W_t^{(q)} \frac{d}{dt} \left(W_t^{(q)}\right)^{-1} V_{t,\dots,t}$$

and the inverse of $\frac{\mathbb{D}^{(q)}}{d\cdot}$ is

$$\left(\frac{\mathbb{D}^{(q)}}{d\cdot}\right)^{-1} = W_{\cdot}^{(q)} \int_0^{\cdot} W_r^{(q)}(\mathrm{ev}_r -)\, dr = \mathcal{W}^q$$

where ev_r, generically, denotes the evaluation operator at r. Furthermore let $\mathcal{R}: \bigwedge^2 TM \to \bigwedge^2 TM$ be the curvature operator. Then the second Weitzenbock curvature $\mathcal{R}^2$ is given by

$$\mathcal{R}^2 = d\bigwedge^2\big(\mathrm{Ric}^{\#}\big) - 2\mathcal{R}.$$

Here the operator $d\bigwedge^2(\mathrm{Ric}^{\#})$, also $(d\bigwedge^2)(\frac{\mathbb{D}}{d\cdot})$ below, is defined using formula (2.3). Therefore using (2.2), for $V \in \bigwedge^2 T_\sigma C_{x_0} M$,

$$\frac{\mathbb{D}^{(2)}}{dt} V_{t,t} = \left(\left(\big(d\bigwedge^2\big)\left(\frac{\mathbb{D}}{d\cdot}\right)\right)V\right)_{t,t} - \mathcal{R}(V_{t,t}), \tag{4.12}$$

whenever all the terms involved make sense. In the above we have identified $\frac{D}{dt}V_{t,t}$ with $(d\bigwedge^2 \frac{D}{d\cdot})(V)_{t,t}$ where the first refers to covariant differentiation of the 2-vector field $\{V_{t,t}: 0 \leqslant t \leqslant T\}$ along σ obtained from the element V in $\bigwedge^2 T_\sigma C_{x_0}$.

C. In this section we shall discuss a system of equations related to the conditional expectation of the Itô map. First note that the curvature operator $\mathcal{R}$ on the manifold M induces a linear map Q_σ on $\bigwedge_\epsilon^2 T_\sigma C_{x_0}$ given by

$$Q_\sigma(G)_{s,t} = \big(\mathbf{1} \otimes W_t^s\big) W_s^{(2)} \int_0^s \big(W_r^{(2)}\big)^{-1}\big(\mathcal{R}_{\sigma_r}(G_{r,r})\big)\, dr, \quad s \leqslant t. \tag{4.13}$$

Equivalently,

$$Q_\sigma(G)_{s,t} = (W_s \otimes W_t)\left(\bigwedge^2\big(W_{\cdot}^{-1}\big) W_{\cdot}^{(2)} \int_0^{\cdot} \big(W_r^{(2)}\big)^{-1}\big(\mathcal{R}_{\sigma_r}(G_{r,r})\big)\, dr\right)_{\min(s,t)}.$$

Clearly

$$\begin{cases} \left(\mathbf{1} \otimes \dfrac{\mathbb{D}}{dt}\right) Q(G)_{s,t} = 0, \quad s < t, \\ \dfrac{\mathbb{D}^{(2)}}{ds} Q(G)_{s,s} = \mathcal{R}(G_{s,s}). \end{cases} \tag{4.14}$$

The second equation is equivalent to $(d\bigwedge^2 \frac{\mathbb{D}}{ds})Q(G)_{s,s} = \mathcal{R}((I+Q)G)_{s,s}$. Define $j_G : [0,T] \to T_{x_0}M \otimes T_{x_0}M$ by

$$j_G(s) = \left(W_s^{-1} \otimes W_s^{-1}\right) W_s^{(2)} \int_0^s \left(W_r^{(2)}\right)^{-1} \left(\mathcal{R}_{\sigma_r}(G_{r,r})\right) dr. \tag{4.15}$$

Then j_G is C^1 and, writing $s \wedge t$ for $\min\{s,t\}$,

$$\left(W_s^{-1} \otimes W_t^{-1}\right) Q_\sigma(G)_{s,t} = j_G(s \wedge t). \tag{4.16}$$

If we set

$$D\left(\bigwedge^2 T_\sigma C_{x_0}\right) = \left\{ \begin{array}{l} u \in \bigwedge_\epsilon^2 T_\sigma C_{x_0} \text{ such that} \\ (1) \text{ for each } 0 \leqslant s < T,\ t \mapsto (/\!/_s^{-1} \otimes /\!/_t^{-1}) u_{s,t} \text{ is} \\ \qquad \text{absolutely continuous on } (s,T]; \\ (2)\ r \mapsto \bigwedge^2 (/\!/_r^{-1}) u_{r,r} \text{ is absolutely continuous on } [0,T] \end{array} \right\}$$

then $Q(G)$ clearly lies in $D(\bigwedge^2 T_\sigma C_{x_0})$. There is another linear map $\mathbb{R}$ on $\bigwedge_\epsilon^2 T_\sigma C_{x_0}$ defined by

$$\mathbb{R}(Z)_{s,t} = (W_s \otimes W_t) \int_0^s \left(\bigwedge^2 W_r^{-1}\right) \left(\mathcal{R}_{\sigma_r}(Z_{r,r})\right) dr, \quad s \leqslant t, \tag{4.17}$$

which also sends $\bigwedge_\epsilon^2 T_\sigma C_{x_0} M$ to $D(\bigwedge^2 T_\sigma C_{x_0} M)$. Furthermore, from Eq. (4.12)

$$\begin{cases} \left(\mathbf{1} \otimes \dfrac{\mathbb{D}}{dt}\right) \mathbb{R}(Z)_{s,t} = 0, \quad s < t, \\ \dfrac{\mathbb{D}^{(2)}}{ds} \mathbb{R}(Z)_{s,s} = \mathcal{R}_{\sigma_s}\left(Z_{s,s} - \mathbb{R}(Z)_{s,s}\right). \end{cases} \tag{4.18}$$

In fact $\mathbf{1} + Q$ and $\mathbf{1} - \mathbb{R}$ are inverse of each other as described in the following lemma. It will be shown later, Section 9D, that $\mathbb{R}$ restricted to $\bigwedge^2 \mathcal{H}^1$ is the curvature operator of the damped Markovian connection on $\mathcal{H}^1$ which is induced by the map $\frac{\mathbb{D}}{d\cdot}$ from the pointwise connection on the L^2 tangent bundle $L^2 T C_{x_0}$.

Lemma 4.2. (i) *Given $G \in D(\bigwedge^2 T_\sigma C_{x_0})$, there is a unique solution $Z \in D(\bigwedge^2 T_\sigma C_{x_0})$ to the following equations*

$$\begin{cases} \left(\mathbf{1} \otimes \dfrac{\mathbb{D}}{dt}\right) Z_{s,t} = \left(\mathbf{1} \otimes \dfrac{\mathbb{D}}{dt}\right) G_{s,t}, \quad s < t, \\ \dfrac{\mathbb{D}^{(2)}}{ds} Z_{s,s} = \left(\left(\left(d\bigwedge^2\right)\left(\dfrac{\mathbb{D}}{d\cdot}\right)\right) G\right)_{s,s}, \\ Z_{0,0} = G_{0,0}. \end{cases} \tag{4.19}$$

The solution is

$$Z_{s,t} = G_{s,t} + Q_\sigma(G)_{s,t}.$$

Conversely for each $Z \in D(\bigwedge^2 T_\sigma C_{x_0})$ *the unique solution to* (4.19) *is given by*

$$G = Z - \mathbb{R}(Z). \tag{4.20}$$

(ii) *As operators on* $\bigwedge^2_\epsilon T_\sigma C_{x_0} M$ *both* Q *and* $\mathbb{R}$ *are compact and* $\mathbf{1}+Q$ *and* $\mathbf{1}-\mathbb{R}$ *are mutual inverses. In particular for all* v *in* $\bigwedge^2_\epsilon T_\sigma C_{x_0} M$,

$$Q(v) = \mathbb{R}\big(v + Q(v)\big),$$
$$Q(\mathbf{1}+Q)^{-1} v = \mathbb{R}(v).$$

(iii) *The following holds on* $D(\bigwedge^2 T_\sigma C_{x_0})$:

$$\big(\textstyle\bigwedge^2 W^{-1} Z\big)_{s,t} - \big(\bigwedge^2 W^{-1} Z\big)_{s\wedge t} = \big(\bigwedge^2 W^{-1} G\big)_{s,t} - \big(\bigwedge^2 W^{-1} G\big)_{s\wedge t}, \tag{4.21}$$

which is equivalent to, for $r \leqslant s \leqslant t$,

$$Z_{r,t} - \big(\mathbf{1}\otimes W_t^s\big) Z_{r,s} = G_{r,t} - \big(\mathbf{1}\otimes W_t^s\big) Z_{r,s}.$$

Proof. Given $G \in D(\bigwedge^2 T_\sigma C_{x_0})$, $Z = (\mathbf{1} + Q_\sigma)(G)$ certainly solves (4.19). For uniqueness let Z be any solution in $D(\bigwedge^2 T_\sigma C_{x_0})$. Solve the first equation in (4.19) to get

$$Z_{s,t} = G_{s,t} + (W_s \otimes W_t)\big(\tilde{j}(s)\big), \quad s \leqslant t, \tag{4.22}$$

some $\tilde{j}(s) \in \bigwedge^2 T_{\sigma_0} M$. Then

$$Z_{s,s} = G_{s,s} + (W_s \otimes W_s)\big(\tilde{j}(s)\big). \tag{4.23}$$

In particular $(W_s \otimes W_s)(\tilde{j}(s))$ is absolutely continuous in s. Substitute the above equation (4.23) into (4.19) and use (4.12) to see

$$\frac{\mathbb{D}^{(2)}}{ds}(W_s \otimes W_s)\big(\tilde{j}(s)\big) = \mathcal{R}(G_{s,s}),$$

giving

$$(W_s \otimes W_s)\big(\tilde{j}(s)\big) = W_s^{(2)} \int_0^s \big(W_r^{(2)}\big)^{-1}\big(\mathcal{R}_{\sigma_r}(G_{r,r})\big)\, dr.$$

Thus $\tilde{j}(s) = j_G(s)$ and uniqueness holds by formula (4.16).

Similarly given $Z \in D(\bigwedge^2 T_\sigma C_{x_0})$, $(\mathbf{1} - \mathbb{R})(Z)$ is seen to satisfy (4.19) given $Z \in D(\bigwedge^2 T_\sigma C_{x_0})$.

Now using the isometry between $\bigotimes_\epsilon^2 C_0T_{x_0}M$ and $C_0([0,T]^2; \bigotimes^2 T_{x_0}M)$ and the Arzèla–Ascoli theorem applied to $(s,t) \mapsto j(s,t)$ for a bounded set of G, we see that $Q : \bigwedge_\epsilon^2 T_\sigma C_{x_0}M \to \bigwedge_\epsilon^2 T_\sigma C_{x_0}M$ is compact. Therefore $\mathbf{1} + Q$ has closed range. Since we have just seen that its range contains all Z in the dense subspace $D(\bigwedge^2 T_\sigma C_{x_0})$ it is surjective and so an isomorphism. By Eq. (4.20) its inverse is $\mathbf{1} - \mathbb{R}$ and so $\mathbb{R}$ is compact. The rest of parts (i) and (ii) follows directly.

Part (iii) follows from (4.22) and (4.23). □

See Section 6 below for a more detailed examination of $Q(V)$.

D. The following theorem gives alternative descriptions of the space $\mathcal{H}_\sigma^2$.

Theorem 4.3. *For any $h^1, h^2 \in L_0^{2,1}\mathbf{R}^m$, set $\underline{h} = h^1 \wedge h^2$. Then*

$$\overline{\bigwedge^2 T\mathcal{I}}(\underline{h}) = (\mathbf{1} + Q)\bigwedge^2 \overline{T\mathcal{I}}(\underline{h}). \tag{4.24}$$

In particular the space $\mathcal{H}_\sigma^2 = \{\overline{\bigwedge^2 T\mathcal{I}_\sigma}(h),\ h \in \bigwedge^2 H\}$ can be characterised by any one of the following:

(i)
$$\mathcal{H}_\sigma^2 = \left\{ \begin{array}{l} u \in D(\bigwedge^2 T_\sigma C_{x_0}),\ \text{such that there exists } G \in \mathcal{H}_\sigma^1 \wedge \mathcal{H}_\sigma^1, \\ \quad \text{with } ((\mathbf{1} \otimes \frac{\mathbb{D}}{d\cdot})u)_{s,t} = ((\mathbf{1} \otimes \frac{\mathbb{D}}{d\cdot})G)_{s,t},\ s < t,\ \text{and} \\ \quad \frac{\mathbb{D}^{(2)}}{ds} u_{s,s} = (((d\bigwedge^2)\frac{\mathbb{D}}{d\cdot})G)_{s,s},\ 0 \leqslant s \leqslant T \end{array} \right\}.$$

(ii)
$$\mathcal{H}_\sigma^2 = \left\{u \in \textstyle\bigwedge_\varepsilon^2 T_\sigma C_{x_0},\ \text{such that } u = v + Q_\sigma(v),\ \text{some } v \in \mathcal{H}_\sigma^1 \wedge \mathcal{H}_\sigma^1\right\},$$

and for $v_1, v_2 \in \bigwedge^2 \mathcal{H}_\sigma^1$, by definition,

$$\left\langle v^1 + Q_\sigma\left(v^1\right), v^2 + Q_\sigma\left(v^2\right)\right\rangle_{\mathcal{H}_\sigma^2} = \left\langle v^1, v^2\right\rangle_{\bigwedge^2 \mathcal{H}_\sigma^1}. \tag{4.25}$$

(iii) *$u \in \mathcal{H}^2$ if and only if $u - \mathbb{R}(u) \in \bigwedge^2 \mathcal{H}^1$. If so*

$$\|u\|_{\mathcal{H}^2} = \left\|u - \mathbb{R}(u)\right\|_{\bigwedge^2 \mathcal{H}^1}.$$

In particular $\mathcal{H}_\sigma^2$ depends on the Riemannian structure of M but not *the choice of stochastic differential equation* (3.2) *provided its LeJan–Watanabe connection in the sense of Elworthy, LeJan and Li* [29] *is the Levi-Civita connection.*

Proof. For $h^1 \wedge h^2 \in L_0^{2,1}(R^m)$, write $V^1 \wedge V^2 = (\bigwedge^2 T\mathcal{I})(h^1 \wedge h^2)$. Then applying Itô's formula in t for $0 \leqslant s < t \leqslant T$ with D_t referring to covariant stochastic differentiation in t,

$$\begin{aligned}
&(\mathbf{1} \otimes D_t)\left(V^1 \wedge V^2\right)_{s,t} \\
&\quad = \frac{1}{2} V_s^1 \otimes \left(\nabla X\left(V_t^2\right) \circ dB_t + X(x_t)\left(\dot{h}_t^2\right) dt\right) \\
&\qquad - \frac{1}{2} V_s^2 \otimes \left(\nabla X\left(V_t^1\right) \circ dB_t + X(x_t)\left(\dot{h}_t^1\right) dt\right) \\
&\quad = \frac{1}{2} V_s^1 \otimes \left(\nabla X\left(V_t^2\right) /\!/_t\, d\beta_t - \frac{1}{2} \mathrm{Ric}^\#\left(V_t^2\right) dt + X(x_t)\left(\dot{h}_t^2\right) dt\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{2} V_s^2 \otimes \left(\nabla X\big(V_t^1\big) /\!/_t \, d\beta_t - \frac{1}{2} \operatorname{Ric}^\#\big(V_t^1\big)\, dt + X(x_t)\big(\dot h_t^1\big)\, dt \right) \\
&= \big(\mathbf{1} \otimes \nabla X(-) /\!/_t \, d\beta_t\big)\big(V^1 \wedge V^2\big)_{s,t} - \left(\mathbf{1} \otimes \frac{1}{2} \operatorname{Ric}^\#(-)\right)\big(V^1 \wedge V^2\big)_{s,t}\, dt \\
&\quad + \frac{1}{2}\big(V_s^1 \otimes X(x_t)\big(\dot h_t^2\big) - V_s^2 \otimes X(x_t)\big(\dot h_t^1\big)\big)\, dt.
\end{aligned}
$$

Write $\overline{V^1 \wedge V^2}$ for the conditional expectation of $V^1 \wedge V^2$ with respect to $\mathcal{F}^{x_0}$, and similarly let $\overline{V}^i$ stand for the conditional expectation of V^i with respect to $\mathcal{F}^{x_0}$. Then by (3.6), following Elworthy and Yor [38], and (4.4)

$$
\begin{aligned}
(\mathbf{1} \otimes D_t)\big(\overline{V^1 \wedge V^2}\big)_{s,t} &= -\left(\mathbf{1} \otimes \frac{1}{2} \operatorname{Ric}^\#(-)\right)\big(\overline{V^1 \wedge V^2}\big)_{s,t}\, dt \\
&\quad + \frac{1}{2}\big(\overline{V}_s^1 \otimes X\big(\dot h_t^2\big) - \overline{V}_s^2 \otimes X\big(\dot h_t^1\big)\big)\, dt.
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
\left(\mathbf{1} \otimes \frac{\mathbb{D}}{d\cdot}\right)\big(\overline{V^1 \wedge V^2}\big)_{s,t} &= \frac{1}{2}\big(\overline{V}_s^1 \otimes X\big(\dot h_t^2\big) - \overline{V}_s^2 \otimes X\big(\dot h_t^1\big)\big) \\
&= \left(\mathbf{1} \otimes \frac{\mathbb{D}}{d\cdot}\right)\big(\overline{V}^1 \wedge \overline{V}^2\big)_{s,t}.
\end{aligned}
$$

On the other hand, Itô's formula, applied to the 2-vector field $\{V_t^1 \wedge V_t^2,\ 0 \leqslant t \leqslant T\}$ along σ in M, gives

$$
\begin{aligned}
D_t\big(V_t^1 \wedge V_t^2\big) &= V_t^1 \wedge \big(\nabla X\big(V_t^2\big) \circ dB_t + X(x_t)\big(\dot h_t^2\big)\, dt\big) \\
&\quad + \big(\nabla X\big(V_t^1\big) \circ dB_t + X(x_t)\big(\dot h_t^1\big)\, dt\big) \wedge V_t^2.
\end{aligned}
$$

Change to Itô differentials and decompose the noise recalling that ∇X vanishes on $[\ker X]^\perp$:

$$
\begin{aligned}
D_t\big(V_t^1 \wedge V_t^2\big) &= V_t^1 \wedge \left(\nabla X\big(V_t^2\big)(/\!/_t \, d\beta_t) - \frac{1}{2} \operatorname{Ric}^\#\big(V_t^2\big)\, dt + X(x_t)\big(\dot h_t^2\big)\, dt\right) \\
&\quad + \left(\nabla X\big(V_t^1\big)(/\!/_t \, d\beta_t) - \frac{1}{2} \operatorname{Ric}^\#\big(V_t^1\big)\, dt + X(x_t)\big(\dot h_t^1\big)\, dt\right) \wedge V_t^2 \\
&\quad + \frac{1}{2} \sum_{i=1}^m \big(\nabla X^i \wedge \nabla X^i\big)\big(V_t^1 \wedge V_t^2\big) \\
&= \big(d{\textstyle\bigwedge}^2\big(\nabla X(-)(/\!/_t \, d\beta_t)\big)\big)\big(V_t^1 \wedge V_t^2\big) \\
&\quad - \left(d{\textstyle\bigwedge}^2\left(\frac{1}{2} \operatorname{Ric}^\#(-)\right)\right)\big(V_t^1 \wedge V_t^2\big)\, dt + \sum_{i=1}^m \big(\nabla X^i \wedge \nabla X^i\big)\big(V_t^1 \wedge V_t^2\big) \\
&\quad + \big(V_t^1 \wedge X(x_t)\big(\dot h_t^2\big) + X(x_t)\big(\dot h_t^1\big) \wedge V_t^2\big)\, dt.
\end{aligned}
$$

But

$$-\left(d\bigwedge^2\left(\frac{1}{2}\operatorname{Ric}^\#(-)\right)\right)+\sum_{i=1}^m \nabla X^i\wedge\nabla X^i=-\frac{1}{2}\mathcal{R}^2, \tag{4.26}$$

as in Elworthy [27] for gradient systems, see also Elworthy, LeJan and Li [30] for the general situation. Again use the technique of Elworthy and Yor [38], taking conditional expectations to get

$$\frac{D}{dt}\overline{V_t^1\wedge V_t^2}=-\frac{1}{2}\mathcal{R}^2\big(\overline{V_t^1\wedge V_t^2}\big)+\overline{V}_t^1\wedge X(x_t)\big(\dot h_t^2\big)+X(x_t)\big(\dot h_t^1\big)\wedge\overline{V}_t^2.$$

Thus

$$\begin{aligned}\frac{\mathbb{D}^{(2)}}{dt}\overline{V_t^1\wedge V_t^2}&=\overline{V}_t^1\wedge X(x_t)\big(\dot h_t^2\big)+X(x_t)\big(\dot h_t^1\big)\wedge\overline{V}_t^2\\&=\left(d\bigwedge^2\left(\frac{\mathbb{D}}{dt}\right)\right)\big(\overline{V}^1\wedge\overline{V}^2\big)_{t,t}.\end{aligned}$$

We have shown given $u=\overline{\bigwedge^2 T\mathcal{I}}(h^1\wedge h^2)$, it is related to $\overline{T\mathcal{I}}(h^1)\wedge\overline{T\mathcal{I}}(h^2)$ by Eq. (4.19). Solve the equation to obtain

$$\overline{\bigwedge^2 T\mathcal{I}}(\underline{h})_{s,t}=\bigwedge^2\overline{T\mathcal{I}}(\underline{h})_{s,t}+\big(\mathbf{1}\otimes W_t^s\big)W_s^{(2)}\int_0^s\big(W_r^{(2)}\big)^{-1}\big(\mathcal{R}\big(\bigwedge^2\overline{T\mathcal{I}}(\underline{h})_{r,r}\big)\big)\,dr,$$

that is, the desired identity (4.24). On the other hand, given u satisfying (4.19) for $G=\bigwedge^2\overline{T\mathcal{I}}(\underline{h})$, $\underline{h}\in\bigwedge^2 L_0^{2,1}(\mathbf{R}^m)$, then $u=\overline{\bigwedge^2 T\mathcal{I}}(\underline{h})$ by uniqueness of the solution. This proves the first equivalence. The second equivalence follows from Lemma 4.2. Part (iii) follows straightforwardly from the previous lemma. □

5. H-one-forms: Exterior differentiation and Hodge decomposition

A. *Differentiation of functions.* For scalar analysis in our context and with this notation, we refer to [35] or for the basic facts to [30]. As emphasised in [35] it is necessary to fix an initial domain, $\operatorname{Dom}(d_{\mathcal{H}})\subset L^2(C_{x_0}M;\mathbf{R})$ for the H-derivative operator $d_{\mathcal{H}}$. We shall choose this to be a subspace which contains the smooth cylindrical functions and consists of BC^2 functions in the Fréchet sense, using the natural Finsler structure of $C_{x_0}M$, see [37]. For example the space of all smooth cylindrical functions. (We will require two derivatives in order to be able to prove that exact H-one-forms are closed.) It is standard, going back to Driver [20], that then $d_{\mathcal{H}}:\operatorname{Dom}(d_{\mathcal{H}})\subset L^2(C_{x_0}M)\to L^2\Gamma\mathcal{H}^*$ is closable. We will denote its closure by $\bar d^0$ to show it is acting on zero forms, or simply by $\bar d$, and let $\mathbb{D}^{2,1}$ be its domain with graph norm. There is the analogous result for functions with values in a separable Hilbert space G. In this case the domain will be written as $\mathbb{D}^{2,1}(G)$ or $\mathbb{D}^{2,1}(C_{x_0}M;G)$ and for almost all $\sigma\in C_{x_0}M$ the derivative $\bar d f_\sigma$ of f at the path σ will be in the space of Hilbert–Schmidt maps $\mathcal{L}_2(\mathcal{H}_\sigma;G)$. As usual for real-valued functions there is the corresponding gradient operator $\nabla:\mathbb{D}^{2,1}\to L^2\Gamma\mathcal{H}$. The negative of its adjoint we write as

$$\mathrm{div} : \mathrm{Dom}(\mathrm{div}) \subset L^2 \Gamma \mathcal{H} \to L^2(C_{x_0}M; \mathbf{R}),$$

so if V is an H-vector field in Dom(div) and $f \in \mathbb{D}^{2,1}$ then

$$\int_{C_{x_0}M} \bar{d}f(V)\,d\mu = \int_{C_{x_0}M} \big\langle \nabla(f)(\sigma), V(\sigma)\big\rangle_{\mathcal{H}_\sigma} d\mu(\sigma)$$

$$= -\int_{C_{x_0}M} f(\sigma)\,\mathrm{div}(V)(\sigma)\,d\mu(\sigma). \tag{5.1}$$

This divergence operator is closed and the standard Riesz correspondence $\phi \mapsto \phi^\#$ with inverse $V \mapsto V^\#$ between H-one-forms and H-vector fields maps the domain of the adjoint d^* of $\bar{d}$ to that of the divergence with $d^*\phi = -\,\mathrm{div}(\phi^\#)$.

For $1 \leqslant p < \infty$ there are the spaces $\mathbb{D}^{p,1}$ defined in the same way as for $p = 2$ but using L^p norms. Spaces of "weakly differentiable" functions $\mathbb{W}^{p,1}(C_{x_0}M; G)$, $1 \leqslant p < \infty$, were also given in [35], loosely following [23]. Here we shall also denote those weak derivatives by $\bar{d}$. Whether $\mathbb{W}^{p,1} = \mathbb{D}^{p,1}$, as occurs on $C_0\mathbf{R}^m$, is an open question. We note the following from [35], cf. [32]. Parts (a) and (b) are essentially equivalent and (a) is a vital step in the proof of the closability of the exterior derivative used below.

Theorem 5.1.

(a) *The map $\overline{T\mathcal{I}(-)}_{.}$ from $L^2(C_0\mathbf{R}^m; H)$ to vector fields on $C_{x_0}M$ given by*

$$\overline{T\mathcal{I}(V)}_\sigma = \mathbb{E}\big\{\omega \mapsto T\mathcal{I}_\omega\big(V(\omega)\big)\big|\mathcal{I}(\omega) = \sigma\big\} \tag{5.2}$$

gives a continuous linear map $\overline{T\mathcal{I}(-)}_{.} : L^2(C_0\mathbf{R}^m; H) \to L^2\Gamma\mathcal{H}$.

(b) *The pull back operation $\phi \mapsto \mathcal{I}^*(\phi)$ defined from one-forms on $C_{x_0}M$ to H-one-forms on $C_0\mathbf{R}^m$ by $(\mathcal{I}^*\phi)_\omega = \phi_{\mathcal{I}(\omega)} \circ T_\omega\mathcal{I}$ extends to give a continuous linear map $\mathcal{I}^* : L^2\Gamma\mathcal{H}^* \to L^2(C_0\mathbf{R}^m; H^*)$.*

(c) *If $f \in \mathbb{D}^{p,1}(C_{x_0}M; G)$ then the composition $f \circ \mathcal{I}$ is in $\mathbb{D}^{p,1}(C_0\mathbf{R}^m; G)$ and then $\bar{d}(f \circ \mathcal{I}) = \mathcal{I}^*(\bar{d}f)$.*

(d) *A measurable function $f : C_{x_0}M \to G$ has $f \in \mathbb{W}^{p,1}(C_{x_0}M; G)$ iff the composition $f \circ \mathcal{I}$ is in $\mathbb{D}^{p,1}(C_0\mathbf{R}^m; G)$ and then the weak derivative $\bar{d}f$ satisfies $\bar{d}(f \circ \mathcal{I}) = \mathcal{I}^*(\bar{d}f)$.*

B. *Exterior differentiation of H-one-forms*. For any C^1 one-form ϕ on $C_{x_0}M$ there is the usual exterior derivative $d\phi$ given by formula (1.2). This can be restricted to give an H-2-form, $d^1_{\mathcal{H}}\phi$ say. Thus $d^1_{\mathcal{H}}\phi_\sigma$ is the composition of $d\phi_\sigma$ with the, continuous, inclusion of $\mathcal{H}^2_\sigma$ in $\bigwedge^2 T_\sigma C_{x_0}M$. As for functions we choose an initial domain $\mathrm{Dom}(d^1_{\mathcal{H}})$ to give an operator

$$d^1_{\mathcal{H}} : \mathrm{Dom}\big(d^1_{\mathcal{H}}\big) \subset L^2\Gamma\big(\mathcal{H}^1\big)^* \to L^2\Gamma\big(\mathcal{H}^2\big)^*.$$

The domain must consist of C^2 one-forms ϕ on $C_{x_0}M$ which satisfy

(i) as an H-one-form, $\phi \in L^\infty \Gamma \mathcal{H}^*$.
(ii) The exterior derivative $d\phi$ when restricted to $\mathcal{H}^2$ is essentially bounded, i.e. $d^1_{\mathcal{H}}\phi \in L^\infty \Gamma \mathcal{H}^{2*}$.
(iii) (Module structure) If $f \in \mathrm{Dom}(d_{\mathcal{H}})$ and $\phi \in \mathrm{Dom}(d^1_{\mathcal{H}})$ then $f\phi \in \mathrm{Dom}(d^1_{\mathcal{H}})$.
(iv) The domain of $d_{\mathcal{H}}$ is mapped into the domain of $d^1_{\mathcal{H}}$ by $d_{\mathcal{H}}$.

All these hold if we use smooth cylindrical functions and forms as initial domains, or C^2 functions and C^1 forms which are bounded together with their exterior derivative using the natural Finsler metric on $C_{x_0}M$. In fact it is shown in [35] that $\mathbb{D}^{2,1}$ is independent of the choice of $\mathrm{Dom}(d_{\mathcal{H}})$ under these restrictions, so we may as well assume that the latter is the space of smooth cylindrical functions.

Under these assumptions we have

Theorem 5.2. *(See [32].) The exterior derivative considered as an operator*

$$d^1_{\mathcal{H}} : \mathrm{Dom}\big(d^1_{\mathcal{H}}\big) \subset L^2\Gamma\big(\mathcal{H}^1\big)^* \to L^2\Gamma\big(\mathcal{H}^2\big)^*$$

is closable.

Since the proof was given in full in [32] and the analogous proof for two-forms is in Part II it will be omitted here. However we note that the main step is to obtain a simple integration by parts formula for elements of $\mathrm{Dom}(d^1_{\mathcal{H}})$ by considering their pull backs, and that of their exterior derivatives to Wiener space by the Itô map. The pull back operation commutes with exterior differentiation, and a simple integration by parts formula for Wiener space can be applied to give the standard closability argument when combined with part (a) of Theorem 5.1. The crucial point is that, for $h \in \bigwedge^2 H$,

$$\int_{C_{x_0}M} d^1_{\mathcal{H}}\phi\big(\overline{\textstyle\bigwedge^2(T\mathcal{I})(h)}\big)\, d\mu_{x_0} = \int_{C_0\mathbf{R}^m} d^1_{\mathcal{H}}\phi\big(\textstyle\bigwedge^2(T\mathcal{I})(h)\big)\, d\mathbb{P}.$$

Let $\bar{d}^1$ denote the closure of $d^1_{\mathcal{H}}$.

Theorem 5.3. *(See [32].) The derivative $\bar{d}^0 f$ of any function $f \in \mathbb{D}^{2,1}$ lies in the domain of $\bar{d}^1$ and*

$$\bar{d}^1\bar{d}^0 f = 0.$$

The derivation property $\bar{d}^1(f\phi) = f\bar{d}^1\phi + \bar{d}^0 f \wedge \phi$ is given meaning and proved in Theorem 7.1 below.

C. *The first L^2 de Rham cohomology group and a Hodge decomposition for H-one-forms.* From the results above we can define the first L^2-cohomology group of $C_{x_0}M$ to be the quotient of the kernel of $\bar{d}^1$ by the image of $\bar{d}^0$. An important result here is due to Fang:

Theorem 5.4. *(See Fang [39].) The image of $\bar{d}^0$ is a closed subspace of $L^2\Gamma\mathcal{H}^*$.*

It is almost a formality now to define the self-adjoint Hodge–Kodaira operator $\triangle$ or $\triangle^1$ by

$$\triangle^1 = \bar{d}^{1*}\bar{d}^1 + \bar{d}^0\bar{d}^{0*}$$

and to obtain the Hodge decomposition. For the details we refer to [32] or Part II.

Theorem 5.5. *(See [32].) There is the orthogonal decomposition*

$$L^2\Gamma\mathcal{H} = \mathrm{Image}(\bar{d}^0) + \overline{\mathrm{Image}}(\bar{d}^{1*}) + \ker\triangle^1$$

where $\overline{\mathrm{Image}(\bar{d}^{1*})}$ *denotes the closure of the image of the adjoint of* $\bar{d}^1$.

6. Tensor products as operators: Algebraic operations on H-one-forms

To show that the exterior product of H-one-forms can be defined as an H-two-form (by a pointwise construction) and to obtain a better understanding of the spaces $\mathcal{H}^2_\sigma$ we will give an interpretation of H-two-vectors in terms of linear maps from $\mathcal{H}^1_\sigma$ to itself. We will also give an example on flat linear Wiener space to show how a theory of tangent processes would lead to analogues of the elements in $\mathcal{H}^2_\sigma$.

A. First we establish our notation and review the well-known results identifying various completions of the algebraic tensor product $H \otimes_0 H$, with spaces of linear maps, and the dualities between the spaces. For example see Ruston [67], though our conventions are slightly different. Here H will be a separable real Hilbert space. Identify $H \otimes_0 H$ with finite rank operators on H by

$$H \otimes_0 H \to \mathcal{L}(H; H)$$

given by

$$(u \otimes v)(h) = \langle v, h\rangle u. \tag{6.1}$$

This extends to an identification of the projective tensor product (the "smallest") $H \otimes_\pi H$ with the space $\mathcal{L}_1(H; H)$ of trace class operators, of our usual $H \otimes H$ with the Hilbert–Schmidt operators $\mathcal{L}_2(H; H)$, and of the inductive, the 'largest reasonable,' completion $H \otimes_\varepsilon H$ with the space of compact operators $\mathcal{L}_c(H; H)$ in $\mathcal{L}(H; H)$:

$$\begin{array}{ccccc}
H \otimes_\pi H & \longrightarrow & H \otimes H & \longrightarrow & H \otimes_\varepsilon H \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\mathcal{L}_1(H; H) & \longrightarrow & \mathcal{L}_2(H; H) & \longrightarrow & \mathcal{L}_c(H; H) \hookrightarrow \mathcal{L}(H; H).
\end{array}$$

The vertical arrows above are isometries, the inner product on $\mathcal{L}_2(H; H)$ being given by

$$\langle S, T\rangle_{\mathcal{L}_2} := \mathrm{trace}\, T^*S = \sum_{i=1}^{\infty}\langle Se_i, Te_i\rangle_H \tag{6.2}$$

for $\{e_i\}_{i=1}^{\infty}$ an orthonormal base of H. So $\operatorname{trace}(u \otimes v) = \langle u, v\rangle$ and

$$\|u \otimes v\|_{\mathcal{L}_2} = \|u\|\|v\| = \|u \otimes v\|_{H\otimes H}.$$

These conventions lead to the following isomorphism with the space of bounded bilinear maps

$$\begin{aligned}\mathcal{L}(H; H) &\to \mathcal{L}(H, H; \mathbf{R}),\\ T &\mapsto \tilde{T}\end{aligned}$$

being given by

$$\tilde{T}(h_1, h_2) = \langle h_1, T h_2\rangle \tag{6.3}$$

with resulting isomorphism, as $\mathcal{L}(H, H; \mathbf{R}) \simeq (H \otimes_\pi H)^*$,

$$\mathcal{L}(H; H) \xrightarrow{D_1} \big(\mathcal{L}_1(H; H)\big)^*$$

expressed by

$$D_1(T)(S) = \operatorname{trace} T^* S. \tag{6.4}$$

This construction shows that D_1 restricts to an isomorphism

$$\mathcal{L}_{skew}(H; H) \xrightarrow{D_1} \big(\textstyle\bigwedge_\pi^2 H\big)^*$$

where $\mathcal{L}_{skew}(H; H)$ refers to the skew adjoint elements of $\mathcal{L}(H; H)$. We shall see later that our operator Q can be considered as a map from $\bigwedge^2 \mathcal{H}^1$ to $\mathcal{L}_{skew}(\mathcal{H}; \mathcal{H})$.

B. We will need the 'double duality' map $\breve{\theta} = D_1^* \circ i$ with i the canonical inclusion $\mathcal{L}_1(H; H) \to \mathcal{L}_1(H; H)^{**}$:

$$\begin{aligned}\mathcal{L}_1(H; H) &\xrightarrow{\breve{\theta}} \mathcal{L}(H; H)^*,\\ \breve{\theta}(T)(S) &:= \operatorname{trace} S^* T,\end{aligned}$$

$T \in \mathcal{L}_1(H; H)$, $S \in \mathcal{L}(H; H)$. Through the isomorphism $\mathcal{L}_1(H; H) \simeq H \otimes_\pi H$, it corresponds to the continuous bilinear map

$$\theta : H \times H \to \mathcal{L}(H; H)^*$$

given by

$$\theta\big(h^1, h^2\big) = \breve{\theta}\big(h^1 \otimes h^2\big)$$

so that

$$\theta\big(h^1, h^2\big)(S) = \big\langle h^1, S h^2\big\rangle. \tag{6.5}$$

C. Let $H = L_0^{2,1}\mathbf{R}^m$. If V belongs to the inductive tensor product $H \otimes_\varepsilon H \hookrightarrow \bigotimes_\varepsilon^2 C_0\mathbf{R}^m$ we see, by taking V primitive, that the corresponding element S^V, say, in $\mathcal{L}(H; H)$ is given by

$$S^V(h)_s \equiv V(h)_s = \int_0^T \left(\frac{\partial}{\partial t} V_{s,t}\right)(\dot{h}_t)\,dt, \tag{6.6}$$

identifying $\frac{\partial}{\partial t} V_{s,t} \in \mathbf{R}^m \otimes \mathbf{R}^m$ with the corresponding element of $\mathcal{L}(\mathbf{R}^m; \mathbf{R}^m)$. For more general kernels $V \in \bigotimes_\varepsilon^2 C_0\mathbf{R}^m$ this can be used to define a linear operator S^V and we let $\mathcal{K}\mathbf{R}^m$ denote the set of such V for which $\frac{\partial}{\partial t} V_{s,t}$ exists for almost all t for each $s \in [0, T]$ and (6.6) determines an element S^V of $\mathcal{L}(H; H)$.

As our main example of an element of $\mathcal{K}\mathbf{R}^m$ let

$$j : [0, T] \to \mathbf{R}^m \otimes \mathbf{R}^m$$

be absolutely continuous with essentially bounded derivative and $j(0) = 0$. Set $V_{s,t} = j(s \wedge t)$. Then V belongs to $\mathcal{K}\mathbf{R}^m$

$$S^V(h)_s = \int_0^T \frac{\partial}{\partial t} j(s \wedge t)(\dot{h}_t)\,dt = \int_0^s j'(r)(\dot{h}_r)\,dr$$

and there is a conjugacy

$$\begin{array}{ccc} L^2([0,T];\mathbf{R}^m) & \xrightarrow{M^{j'}} & L^2([0,T];\mathbf{R}^m) \\ \uparrow \frac{d}{d\cdot} & & \uparrow \frac{d}{d\cdot} \\ L_0^{2,1}\mathbf{R}^m & \xrightarrow{S^V} & L_0^{2,1}\mathbf{R}^m \end{array}$$

to the multiplication (i.e. zero order) operator $M^{j'}$ given by

$$M^{j'}(f)(t) = j'(t) f(t)$$

for $j'(t)$ considered to be in $\mathcal{L}(\mathbf{R}^m; \mathbf{R}^m)$. In particular we see that in general such V do not correspond to compact operators, let alone to elements of $H \otimes H$. Also for $\theta : H \times H \to \mathcal{L}(H; H)^*$ defined in Section 6C we see from (6.5) that

$$\theta\big(h^1, h^2\big)\big(S^V\big) = \int_0^T \big\langle \dot{h}_s^1, j'(s)\big(\dot{h}_s^2\big)\big\rangle_{\mathbf{R}^m}\,ds. \tag{6.7}$$

Theorem 6.1. *For V in $\mathcal{H}_\sigma^1 \wedge \mathcal{H}_\sigma^1$ let $Q(V) \in \bigwedge_\epsilon^2 T_\sigma C_{x_0}$ be defined by* (4.13). *Then considered as a kernel it determines an element $S^{Q(V)}$ of $\mathcal{L}(\mathcal{H}_\sigma^1; \mathcal{H}_\sigma^1)$ which is conjugate to a multiplication operator M on $L^2 T_\sigma C_{x_0} M$:*

$$\begin{array}{ccc} L^2T_\sigma C_{x_0}M & \xrightarrow{M} & L^2T_\sigma C_{x_0}M \\ \uparrow \frac{\mathbb{D}}{\partial t} & & \uparrow \frac{\mathbb{D}}{\partial t} \\ \mathcal{H}^1_\sigma & \xrightarrow{S^{Q(V)}} & \mathcal{H}^1_\sigma. \end{array}$$

Here $M(u)_t = W_t j'_V(t)(W_t^{-1}u_t)$ *for* j_V *given by Eq.* (4.15) (*and so* j'_V *by* (6.9) *below*), $u \in L^2TC_{x_0}M$.

Proof. Set $\tilde{V}_{s,t} = (W_s^{-1} \otimes W_t^{-1})V_{s,t}$. Let $\tilde{Q} : \bigwedge^2 L_0^{2,1}T_{x_0}M \to \bigwedge^2 C_0T_{x_0}M$ be given by

$$\tilde{Q}(U)_{s,t} = \left(W_s^{-1} \otimes W_t^{-1}\right)Q\left(\bigwedge^2(U)\right)_{s,t}. \tag{6.8}$$

Then from Eq. (4.16)

$$\tilde{Q}(\tilde{V})_{s,t} = j_V(s \wedge t).$$

As earlier $S^{\tilde{Q}(\tilde{V})}$ is conjugate, by $\frac{d}{dt}$, to $M^{j'_V}$ acting on $L^2([0,T]; T_{x_0}M)$.

For $h \in \mathcal{H}^1_\sigma$ we have $S^{Q(V)}(h)_t = W_t(S^{\tilde{Q}(\tilde{V})}(W_\cdot^{-1}h_\cdot))_t$ so

$$\begin{aligned} \frac{\mathbb{D}}{dt}S^{Q(V)}(h)_t &= W_t\frac{d}{dt}\left(S^{\tilde{Q}(\tilde{V})}\left(W_\cdot^{-1}h_\cdot\right)\right) = W_t\left(M^{j'_V}\left(\frac{d}{dt}W_\cdot^{-1}h_\cdot\right)\right)_t \\ &= W_t\left(M^{j'_V}\left(W_\cdot^{-1}\frac{\mathbb{D}}{dt}h_\cdot\right)\right)_t = W_t j'_V(t)W_t^{-1}\frac{\mathbb{D}}{dt}h \end{aligned}$$

proving the conjugacy. $\square$

Thus $Q(V)_\sigma$ corresponds to an element of $\mathcal{L}_{skew}(\mathcal{H}_\sigma; \mathcal{H}_\sigma)$, and so of $(\mathcal{H}^*_\sigma \otimes_\pi \mathcal{H}^*_\sigma)^*$, but is not compact and in particular does not belong to $\bigwedge^2 \mathcal{H}^1_\sigma$. This yields

Corollary 6.2. *There is a natural inclusion of* $\mathcal{H}^2_\sigma$ *in* $\mathcal{L}_{skew}(\mathcal{H}_\sigma; \mathcal{H}_\sigma)$ *given by* $V \mapsto S^V$.

Note that by the definition (4.15) and formula (4.12)

$$\begin{aligned} j'_V(t) &= \left(W_t^{-1} \otimes W_t^{-1}\right)\left(\frac{\mathbb{D}^{(2)}}{dt} + \mathcal{R}_{\sigma_t}\right)W_t^{(2)}\int_0^t \left(W_r^{(2)}\right)^{-1}\mathcal{R}_{\sigma_r}\left(\bigwedge^2(W_r)V_{r,r}\right)dr \\ &= \left(W_t^{-1} \otimes W_t^{-1}\right)\left(\mathcal{R}_{\sigma_t}\left(\bigwedge^2(W_t)V_{t,t}\right)\right) \\ &\quad + \left(W_t^{-1} \otimes W_t^{-1}\right)\left(\mathcal{R}_{\sigma_t}W_t^{(2)}\int_0^t \left(W_r^{(2)}\right)^{-1}\mathcal{R}_{\sigma_r}\left(\bigwedge^2(W_r)V_{r,r}\right)dr\right). \end{aligned} \tag{6.9}$$

Remark 6.3. The inclusion can also be seen geometrically from the fact that if $U \in \mathcal{H}^2_\sigma$ then $U - \mathbb{R}(U) \in \bigwedge^2 \mathcal{H}_\sigma \subset \mathcal{L}_{skew}(\mathcal{H}_\sigma; \mathcal{H}_\sigma)$ where $\mathbb{R}$ is the curvature operator of the damped Markovian connection which takes values in $\mathcal{L}_{skew}(\mathcal{H}_\sigma; \mathcal{H}_\sigma)$; see Section 9D below.

D. *Interior and exterior products.* For any separable Hilbert space H define the interior product by an element h of H by

$$\iota_h : H \otimes_0 H \to H, \quad h \in H,$$
$$\iota_h\big(h^1 \otimes h^2\big) := \big\langle h^1, h\big\rangle h^2 = S^*(h),$$

where $S \in \mathcal{L}(H; H)$ corresponds to $h^1 \otimes h^2$. Thus ι_h extends to a continuous linear map over all the completed tensor products we use and even can be defined consistently as

$$\iota_h : \mathcal{L}(H; H) \to H, \quad \text{by}$$
$$\iota_h(S) := S^*(h).$$

E. The first part of the following lemma is standard, but the conventions are important, see Appendix A.

Lemma 6.4. (i) *The maps $\iota_h : H \otimes H \to H$ and $h\otimes : H \to H \otimes H$ are mutually adjoint as are the maps $\iota_h : \bigwedge^2 H \to H$ and $h\wedge : H \to \bigwedge^2 H$.*

(ii) *The adjoint of $h\otimes : H \to H \otimes_\pi H$ is $\iota_h : \mathcal{L}(H; H) \to H$, identifying $(H \otimes_\pi H)^*$ with $\mathcal{L}(H; H)$ by D_1 as in* (6.3). *Similarly the adjoint of $h\wedge : H \to \bigwedge_\pi^2 H$ is the restriction of ι_h to the skew-symmetric elements $\mathcal{L}_{skew}(H; H)$ of $\mathcal{L}(H; H)$, using the restrictions of D_1* (*see Section* 6B *above*).

Proof of (ii). If $S \in \mathcal{L}(H; H)$ and $h_1 \in H$ then

$$\big\langle \iota_h(S), h_1\big\rangle = \big\langle S^*(h), h_1\big\rangle = \operatorname{trace}\big[S^* \circ (h \otimes h_1)\big]$$
$$= D_1(S)(h \otimes h_1) = D_1(S)(h \otimes \cdot)(h_1)$$

while if S is skew-symmetric

$$D_1(S)(h \otimes h_1) = \langle h, Sh_1\rangle = \frac{1}{2}\big\{\langle h, Sh_1\rangle - \langle Sh, h_1\rangle\big\} = D_1(S)(h \wedge h_1). \qquad \square$$

F. Now take $H = L_0^{2,1} T_{x_0} M$ and consider $\tilde{Q} : \bigwedge^2 H \to \bigwedge^2 C_0 T_{x_0} M$ given as in (6.8). The inclusion $H \hookrightarrow C_0 T_{x_0} M$ has an injective adjoint with dense range $(C_0 T_{x_0} M)^* \to H$. Let $\phi^\#$ denote the image of $\phi \in (C_0 T_{x_0} M)^*$ under this map. There is the interior product

$$\iota_\phi : \bigwedge\nolimits^2 C_0 T_{x_0} M \to C_0 T_{x_0} M$$

given by

$$\iota_\phi\big(u^1 \wedge u^2\big) = \frac{1}{2}\big(\phi\big(u^1\big)u^2 - \phi\big(u^2\big)u^1\big).$$

Lemma 6.5. *For $\underline{h} \in \bigwedge^2 H$ consider $S^{\tilde{Q}_\sigma(\underline{h})} \in \mathcal{L}(H; H)$. Then for $\phi \in (C_0 T_{x_0} M)^*$ we have*

$$\iota_\phi\big(\tilde{Q}_\sigma(\underline{h})\big) = \iota_{\phi^\#} S^{\tilde{Q}_\sigma(\underline{h})} = -S^{\tilde{Q}_\sigma(\underline{h})}\big(\phi^\#\big).$$

Proof. Write ϕ in terms of a $T_{x_0}M$-valued countably additive measure, m^ϕ, of finite variation on $[0, T]$ so

$$\phi(w) = \int_0^T \langle w_s, dm^\phi(s)\rangle, \quad w \in C_0T_{x_0}M.$$

Then, if $\underline{u} = u^1 \wedge u^2 \in \bigwedge^2 C_0T_{x_0}M$ has $u_{s,t} = u_{t,s}$,

$$\iota_\phi(\underline{u})_t = \frac{1}{2}\int_0^T \langle u_s^1, dm^\phi(s)\rangle u_t^2 - \langle u_s^2, dm^\phi(s)\rangle u_t^1 = -\int_0^T \underline{u}_{t,s}\big(dm^\phi(s)\big)$$

treating $\underline{u}_{t,s} \in \bigwedge^2 T_{x_0}M$ as an element of $\mathcal{L}_{skew}(T_{x_0}M; T_{x_0}M)$. Thus

$$\big(\iota_\phi\big[\tilde{Q}_\sigma(\underline{h})\big]\big)_t = -\int_0^T j_{\underline{h}}(s\wedge t)\big(dm^\phi(s)\big) = -\int_0^t \left(\frac{d}{ds} j_{\underline{h}}(s)\right)\left(\int_s^T dm^\phi(r)\right) ds. \tag{6.10}$$

On the other hand, if $k \in H$,

$$\int_0^T \langle \dot{\phi}_s^\#, \dot{k}_s\rangle\, ds = \langle \phi^\#, k\rangle_H = \int_0^T \langle k_s, dm^\phi(s)\rangle\, ds = \int_0^T \left\langle \dot{k}_s, \int_s^T dm^\phi(r)\right\rangle ds.$$

Thus $\phi_t^\# = \int_0^t (\int_s^T dm^\phi(r))\, ds$ (a well-known result in Wiener space theory). This, using (6.10) and then Section 6C above, gives

$$\begin{aligned}\big(\iota_\phi\big[\tilde{Q}_\sigma(\underline{h})\big]\big)_t &= -\int_0^t \frac{d}{ds} j_{\underline{h}}(s)\,\big(\dot{\phi}_s^\#\big)\, ds = -S^{\tilde{Q}_\sigma(\underline{h})}\big(\phi^\#\big)\\ &= \iota_{\phi^\#} S^{\tilde{Q}_\sigma(\underline{h})}\end{aligned}$$

by definition (see Section 6E). □

Remark. The same calculation shows that the analogous result holds with general elements of $\mathcal{K}T_{x_0}M$, see Section 6C, replacing $\tilde{Q}_\sigma(h)$.

G. Set

$$\tilde{\mathcal{H}}_\sigma^2 = (1 + \tilde{Q}_\sigma)\big[\textstyle\bigwedge^2 H\big] \subset \bigwedge^2 C_0T_{x_0}M.$$

From Section 6D above we can consider elements of $\tilde{\mathcal{H}}_\sigma^2$ as skew-symmetric bounded linear operators on H. This can be exploited to extend the definition of exterior products:

Lemma 6.6. *The mapping*

$$(C_0 T_{x_0} M)^* \times (C_0 T_{x_0} M)^* \to \big(\tilde{\mathcal{H}}^2_\sigma\big)^*$$

given by

$$\big(\phi^1, \phi^2\big) \to \phi^1 \wedge \phi^2\big|_{\tilde{\mathcal{H}}^2_\sigma}$$

extends to a continuous, antisymmetric, bilinear map

$$H \times H \xrightarrow{\triangle} \big(\tilde{\mathcal{H}}^2_\sigma\big)^*$$

inducing a bounded linear map $\tilde{\theta}_\sigma : \bigwedge^2_\pi H \to (\tilde{\mathcal{H}}^2_\sigma)^*$ *which agrees with the map* $\breve{\theta}$ *of* Section 6C:

$$\begin{tikzcd} & \mathcal{L}(H;H)^* \arrow[d] \\ \bigwedge^2_\pi H \arrow[ur, "\breve{\theta}"] \arrow[r, "\tilde{\theta}_\sigma"] & (\tilde{\mathcal{H}}^2_\sigma)^* \end{tikzcd}$$

using the inclusion of $\tilde{\mathcal{H}}^2_\sigma$ *into* $\mathcal{L}(H; H)$.

Proof. For $S \equiv S^{\tilde{Q}_\sigma(\underline{h})} \in \mathcal{L}_{skew}(H; H)$ corresponding to $\tilde{Q}_\sigma(\underline{h})$ as above, if $\phi^1, \phi^2 \in (C_0 T_{x_0} M)^*$ then using Lemma 6.5,

$$\begin{aligned} \big(\phi^1 \wedge \phi^2\big)\big(\tilde{Q}_\sigma(h)\big) &= \phi^2\big(\iota_{\phi^1}\big(\tilde{Q}_\sigma(\underline{h})\big)\big) = -\phi^2\big(S\big(\phi^{1^\#}\big)\big) \\ &= -\big\langle \phi^{2^\#}, S\big(\phi^{1^\#}\big)\big\rangle_H. \end{aligned} \tag{6.11}$$

Also

$$\|S\|_{\mathcal{L}(H;H)} = \sup_{0 \leqslant s \leqslant T} \big|\alpha_{\underline{h}}(s)\big| \leqslant \text{const} \cdot \sup_r |h_{rr}| \leqslant \text{const} \cdot \|\underline{h}\|_{\bigwedge^2 H} \tag{6.12}$$

for $\alpha_{\underline{h}}$ the multiplication operator corresponding to S as in Section 6C, i.e. $\alpha_{\underline{h}}(t) = \frac{d}{dt} j_{\underline{h}}(t)$ given by Eq. (6.9). Therefore

$$\big|\big\langle \phi^{2^\#}, S\phi^{1^\#}\big\rangle\big| \leqslant \text{const} \cdot \|\underline{h}\|_{\bigwedge^2 H} \cdot \big\|\phi^{2^\#}\big\|_H \cdot \big\|\phi^{1^\#}\big\|_H.$$

This shows we have $\tilde{\theta}_\sigma \in \mathcal{L}(\bigwedge^2_\pi H; (\tilde{\mathcal{H}}^2_\sigma)^*)$. This agrees with $\breve{\theta}$, as required, by equality (6.5). □

H. We now interpret these result in terms of $\mathcal{H}$-forms and $\mathcal{H}$ vectors on $C_{x_0} M$.

Theorem 6.7. (i) *For $v \in \mathcal{H}_\sigma^1$ there is an interior product* (*annihilation operator*)

$$\iota_v : \mathcal{H}_\sigma^2 \to \mathcal{H}_\sigma^1$$

which is continuous linear, and agrees with the usual ι_ϕ for $\phi \in (T_\sigma C_{x_0} M)^*$ when $v = \phi^\#$. The map $(v, U) \mapsto \iota_v(U)$ is in $\mathcal{L}(\mathcal{H}_\sigma^1, \mathcal{H}_\sigma^2; \mathcal{H}_\sigma^1)$ and is bounded uniformly in σ.

(ii) *The map*

$$(T_\sigma C_{x_0} M)^* \times (T_\sigma C_{x_0} M)^* \to \left(\mathcal{H}_\sigma^2\right)^*,$$
$$\left(\phi^1, \phi^2\right) \mapsto \left(\phi^1 \wedge \phi^2\right)\big|_{\mathcal{H}_\sigma^2}$$

extends to give a continuous linear map

$$\lambda_\sigma : \left(\mathcal{H}_\sigma^1\right)^* \wedge_\pi \left(\mathcal{H}_\sigma^1\right)^* \to \left(\mathcal{H}_\sigma^2\right)^*$$

which is bounded uniformly in σ as an element of $\mathcal{L}((\mathcal{H}_\sigma^1)^* \wedge_\pi (\mathcal{H}_\sigma^1)^*; (\mathcal{H}_\sigma^2)^*)$.

(iii) *Moreover, if $v \in \mathcal{H}_\sigma^1$, $\ell \in (\mathcal{H}_\sigma^1)^*$ and $U \in \mathcal{H}_\sigma^2$,*

$$\lambda_\sigma\left(v^\# \wedge \ell\right)(U) = \ell(\iota_v U).$$

Proof. (i) The existence of ι_v and its properties come from Lemma 6.5 and the bounds on S noted in Eq. (6.12).

(ii) Lemma 6.6 provides the proof of (ii) with λ_σ being conjugate by $\bigwedge^2(W.)$ to the map $\tilde{\theta}_\sigma$ of Lemma 6.6. We see from there that $\tilde{\theta}_\sigma$ is bounded uniformly in σ if the inclusion $\mathcal{H}_\sigma^2 \to \mathcal{L}(H; H)$ is. However this is essentially the map $\underline{h} \mapsto S^{\tilde{Q}_\sigma(\underline{h})}$ again.

For (iii) approximate $v^\#$ and ℓ by elements coming from $(T_\sigma C_{x_0})^*$. By Lemma 6.5, if $U = V + Q(V)$

$$\iota_v(U) = \iota_v(V) - S^{Q(V)}\left(v^\#\right)$$

so

$$\begin{aligned}\ell\left(\iota_v(U)\right) &= \ell\left(\iota_v(V)\right) - \left\langle \ell^\#, S^{Q(V)}\left(v^\#\right)\right\rangle_{\mathcal{H}_\sigma^1} \\ &= \left(v^\# \wedge \ell\right)(V) + \left(v^\# \wedge \ell\right)\left(Q_\sigma(V)\right), \quad \text{by (6.11).} \qquad \square\end{aligned}$$

We shall write $\lambda_\sigma(\phi \wedge \psi)$ as $\phi \wedge_\pi \psi$ when no confusion can arise.

Remark 6.8. The map λ_σ is independent of the choice of the Hilbert space inner product given to $\mathcal{H}_\sigma^1$, or $\mathcal{H}_\sigma^2$. Its adjoint gives a continuous map

$$\lambda_\sigma^* : \mathcal{H}_\sigma^2 \to \left(\left(\mathcal{H}_\sigma^1\right)^* \wedge_\pi \left(\mathcal{H}_\sigma^1\right)^*\right)^*$$

of $\mathcal{H}_\sigma^2$ into the skew-symmetric bi-forms on $(\mathcal{H}_\sigma^1)^*$.

7. The derivation property for $\bar{d}^1$

A. We can now formulate and prove the derivation property of $\bar{d}^1$.

Theorem 7.1. *Suppose $f: C_{x_0}M \to \mathbf{R}$ is in* $\mathrm{Dom}(\bar{d}^0)$ *and* $\phi \in \mathrm{Dom}(\bar{d}^1) \cap L^\infty\Gamma(\mathcal{H}^1)^*$ *with* $\bar{d}^1\phi \in L^\infty\Gamma(\mathcal{H}^2)^*$. *Then* $f\phi \in \mathrm{Dom}(\bar{d}^1)$ *and*

$$\bar{d}^1(f\phi) = \bar{d}^0 f \wedge_\pi \phi + f\big(\bar{d}^1\phi\big)$$

where $\wedge_\pi$ is defined above by Theorem 6.7.

Proof. Let $\{\phi_j\}_{j=1}^\infty$ be a sequence in $\mathrm{Dom}(d^1_{\mathcal{H}})$ with $\phi_j \to \phi$ in $L^2\Gamma(\mathcal{H}^1)^*$ and $d^1\phi_j \to \bar{d}^1\phi$ in $L^2\Gamma(\mathcal{H}^2)^*$. Assume first that $f \in \mathrm{Dom}(d_{\mathcal{H}})$. Then $f\phi_j \to f\phi$ in $L^2\Gamma(\mathcal{H}^1)^*$ by the module structure of $\mathrm{Dom}(d^1_{\mathcal{H}})$, and by standard calculus

$$d(f\phi_j) = df \wedge \phi_j + f\, d\phi_j,$$

therefore

$$d(f\phi_j)|_{\mathcal{H}^2_\sigma} = \lambda_\sigma\big(df|_{\mathcal{H}^1_\sigma} \wedge \phi_j|_{\mathcal{H}^1_\sigma}\big) + f(d\phi_j)|_{\mathcal{H}^2_\sigma}$$

in the notation of Theorem 6.7. By the uniform bound on λ_σ from that theorem, and taking a subsequence if necessary to assume $\phi_j|_{\mathcal{H}^1_\sigma} \to \phi|_{\mathcal{H}^1_\sigma}$ for almost all σ, we see

$$\lambda_\sigma\big(df|_{\mathcal{H}^1_\sigma} \wedge \phi_j|_{\mathcal{H}^1_\sigma}\big) \to \lambda_\sigma\big(\bar{d}^0 f_\sigma \wedge \phi_\sigma\big)$$

almost surely and so in L^2 by the dominated convergence theorem. Since $f(d\phi_j) \to f\bar{d}^1\phi$ and $f\phi_j \in \mathrm{Dom}(d^1_{\mathcal{H}})$ the result follows for $f \in \mathrm{Dom}(d_{\mathcal{H}})$.

For general $f \in \mathrm{Dom}(\bar{d}^0)$ take $\{f_j\}_{j=1}^\infty$ in $\mathrm{Dom}(d_{\mathcal{H}})$ with $f_j \to f$ in L^2 and $\bar{d}f_j \to \bar{d}f$ in $L^2\Gamma(\mathcal{H}^1)^*$. From above we know that $f_j\phi \in \mathrm{Dom}(\bar{d}^1)$ with

$$\bar{d}^1(f_j\phi) = \bar{d}f_j \wedge_\pi \phi - f_j\big(\bar{d}^1\phi\big), \quad j = 1 \text{ to } \infty.$$

Now ϕ and $\bar{d}^1\phi$ are bounded so as before we see $\bar{d}f_j \wedge_\pi \phi \to df \wedge_\pi \phi$ and $f_j\bar{d}^1\phi \to f\bar{d}^1\phi$, both in $L^1\Gamma(\mathcal{H}^2)^*$, completing the proof. □

8. Infinitesimal rotations as divergences

We will say that a p-vector field V on $C_{x_0}M$ (or similarly on $C_0(\mathbf{R}^m)$), *has a divergence* if there exists $\operatorname{div} V \in L^1\Gamma\bigwedge^{p-1}TC_{x_0}M$ such that for all smooth, bounded, cylindrical $(p-1)$-forms ϕ we have

$$\int_{C_{x_0}M} d\phi(V)\, d\mu_{x_0} = -\int_{C_{x_0}M} \phi(\operatorname{div} V)\, d\mu_{x_0}. \tag{8.1}$$

For $p=1$ from Driver [20] we know that not only do sufficiently regular elements of $L^2\Gamma\mathcal{H}^1$ have divergences but so do the *infinitesimal rotations* $R^\alpha \in L^2\Gamma \bigwedge^2 TC_{x_0}M$ given by

$$R_t^\alpha = /\!/_t \int_0^t /\!/_s^{-1} \alpha_s \, dx_s \tag{8.2}$$

where $\alpha_s : C_{x_0}M \to \mathcal{L}_{skew}(T_{x_s}M; T_{x_s}M)$, $0 \leqslant s \leqslant T$, is in L^2 and progressively measurable. Indeed

$$\operatorname{div} R_\cdot^\alpha = 0.$$

For more examples of one-vector fields with divergences see Bell [9], Cruzeiro and Malliavin [19], Fang [40], and Hu, Üstünel and Zakai [45] and for p-vector fields see [33]. As in finite dimensions if a p-vector field V has a divergence $\operatorname{div} V$, when $p>1$, then $\operatorname{div} V$ has a vanishing divergence. In view of the looseness of the definition and the homotopical triviality of $C_{x_0}M$ we would expect that a field with a divergence which is zero would necessarily be a divergence, and we will give some evidence for this which also sheds light on the structure of our modified de Rham complex.

First we observe that the exterior product of suitably regular H-vector fields in Dom(div) has a divergence. For this let $V^1, V^2 \in L^2\Gamma\mathcal{H}^1$. Then we have an L^2 section $V^1 \wedge V^2$ of $\mathcal{H}^1 \wedge \mathcal{H}^1$. If ϕ is a smooth (bounded) cylindrical 1-form, then as discussed in Appendix B,

$$2\, d\phi\big(V^1 \wedge V^2\big) = \iota_{V^1} d\iota_{V^2}(\phi) - \iota_{V^2} d\iota_{V^1}(\phi) - 2\phi\big(\big[V^1, V^2\big]\big)$$

provided V^1, V^2 are sufficiently regular. Give such a regularity

$$\begin{aligned} &2 \int_{C_{x_0}M} d\phi\big(V^1 \wedge V^2\big)\, d\mu_{x_0} \\ &\quad = \int_{C_{x_0}M} \iota_{V^1}(\phi) \operatorname{div} V^2 \, d\mu_{x_0} - \int_{C_{x_0}M} \iota_{V^2}(\phi) \operatorname{div} V^1 \, d\mu_{x_0} - \int_{C_{x_0}M} \phi\big(\big[V^1, V^2\big]\big)\, d\mu_{x_0}. \end{aligned}$$

Thus $V^1 \wedge V^2$ has a divergence with

$$2 \operatorname{div}\big(V^1 \wedge V^2\big) = -\big(\operatorname{div} V^2\big) V^1 + \big(\operatorname{div} V^1\big) V^2 + \big[V^1, V^2\big]. \tag{8.3}$$

The first two terms are sections of $\mathcal{H}^1$ but as is well known, Cruzeiro and Malliavin [18], Driver [21], the bracket involves a stochastic integral of the form I for

$$I_t = /\!/_t \int_0^t /\!/_s^{-1} \mathcal{R}\big(V_s^1 \wedge V_s^2\big)\, dx_s, \tag{8.4}$$

i.e. an infinitesimal rotation. The above applies in particular to $V^i = \overline{T\mathcal{I}}(h^i)$ for $h^i \in W^{2,1}(C_{x_0}M; H)$, $i = 1, 2$.

Also if $\underline{h}: C_{x_0}M \to \bigwedge^2 H$ is in $W^{2,1}$, the 2-vector field $\overline{\bigwedge^2 T\mathcal{I}(\underline{h})}$ has a divergence with $\operatorname{div} \overline{\bigwedge^2 T\mathcal{I}(\underline{h})} = \overline{T\mathcal{I}(\operatorname{div}(\underline{h} \circ \mathcal{I}))}$. Indeed for ϕ a smooth cylindrical one-form

$$
\begin{aligned}
\int\limits_{C_{x_0}M} d\phi\big(\overline{\textstyle\bigwedge^2 T\mathcal{I}(\underline{h})}\big)\, d\mu_{x_0} &= \int\limits_{C_0\mathbf{R}^m} \mathcal{I}^*(d\phi)(\underline{h} \circ \mathcal{I})\, dP \\
&= \int\limits_{C_0\mathbf{R}^m} d\big(\mathcal{I}^*\phi\big)(\underline{h} \circ \mathcal{I})\, dP = -\int\limits_{C_0\mathbf{R}^m} \mathcal{I}^*\phi(\operatorname{div} \underline{h} \circ \mathcal{I})\, dP \\
&= -\int\limits_{C_0\mathbf{R}^m} \phi\big(\overline{T\mathcal{I}\big(\operatorname{div}(\underline{h} \circ \mathcal{I})\big)}\big)\, dP.
\end{aligned}
$$

Here we use the fact that since $\underline{h} \in W^{2,1}$, we have $\underline{h} \circ \mathcal{I} \in \mathbb{D}^{2,1} \subset \operatorname{Dom}(\operatorname{div})$. Consequently,

$$
\operatorname{div}\big(\overline{\textstyle\bigwedge^2 T\mathcal{I}(\underline{h})}\big) = \overline{T\mathcal{I}(\operatorname{div} \underline{h})}. \tag{8.5}
$$

(For another version of this result see Section 8E.) On the other hand,

$$
\overline{\textstyle\bigwedge^2 T\mathcal{I}(\underline{h})} = \textstyle\bigwedge^2 \overline{T\mathcal{I}}(\underline{h}) + Q\big(\textstyle\bigwedge^2 \overline{T\mathcal{I}}(\underline{h})\big).
$$

Thus:

Proposition 8.1. *For $\underline{h} = h^1 \wedge h^2$ with $h^i \in W^{2,1}(C_{x_0}M; H)$, $i = 1, 2$, the two-vector field $Q(\bigwedge^2 \overline{T\mathcal{I}}(\underline{h}))$ has a divergence with*

$$
\operatorname{div} Q\big(\textstyle\bigwedge^2 \overline{T\mathcal{I}}(\underline{h})\big) = \overline{T\mathcal{I}(\operatorname{div} \underline{h})} - \operatorname{div}\big(\textstyle\bigwedge^2 \overline{T\mathcal{I}}(\underline{h})\big).
$$

Since $\overline{T\mathcal{I}}(\operatorname{div} \underline{h}) \in \Gamma\mathcal{H}^1$ we see that $\operatorname{div} Q(\bigwedge^2 \overline{T\mathcal{I}}(\underline{h}))$ must cancel out the infinitesimal rotation term I in $\operatorname{div}(\bigwedge^2 \overline{T\mathcal{I}}(\underline{h}))$. A geometrical interpretation of this is given below, see Section 9. The following result concerning the flat Wiener space case shows how this can happen. It should be considered together with formula (4.16) for Q and the discussion in Section 6C.

Proposition 8.2. *Every two-vector field $V : C_0(\mathbf{R}^m) \to \bigwedge^2 C_0(\mathbf{R}^m)$ given by $V_{s,t} = j(s \wedge t)$ for $j(t) = \int_0^t \alpha_r\, dr$, where $\alpha_\cdot : [0, T] \times C_0(\mathbf{R}^m) \to \mathcal{L}_{skew}(\mathbf{R}^m; \mathbf{R}^m)$ is progressively measurable with $\int_{C_0(\mathbf{R}^m)} \int_0^T |\alpha_s|^2\, ds\, d\mu_{x_0} < \infty$, has a divergence. It is given by*

$$
\operatorname{div} V = \int_0^{\cdot} \alpha_s\, dB_s,
$$

i.e. $\operatorname{div} V = R^\alpha$.

Proof. Let $f : C_0(\mathbf{R}^m) \to \mathbf{R}$ be bounded and C^∞ and let $\ell \in C_0(\mathbf{R}^m)^*$. Define the 1-form ϕ on $C_0(\mathbf{R}^m)$ by

$$\phi_\omega(v) = f(\omega)\ell(v).$$

Bounded cylindrical 1-forms can be written as sums of such forms. Then $d\phi = df \wedge \ell$.

Let k be the image of ℓ under the inclusions $C_0(\mathbf{R}^m)^* \to L_0^{2,1}(\mathbf{R}^m)$ adjoint to the inclusion of $L_0^{2,1}$ in C_0.

From (6.7) above we see

$$d\phi(V) = \int_0^T \big\langle \overbrace{(\nabla_H f)}^{\cdot}_s, \alpha_s \dot{k}_s \big\rangle_{\mathbf{R}^m} \, ds = df\left(\int_0^{\cdot} \alpha_s \dot{k}_s \, ds \right).$$

Thus

$$\begin{aligned}
\int_{C_0\mathbf{R}^m} d\phi(V)\, dP &= \int_{C_0\mathbf{R}^m} f(\omega) \int_0^T \langle \alpha_s \dot{k}_s, dB_s \rangle_{\mathbf{R}^m} \, dP(\omega) \\
&= -\int_{C_0\mathbf{R}^m} f(\omega) \int_0^T \langle \dot{k}_s, \alpha_s \, dB_s \rangle_{\mathbf{R}^m} \, dP(\omega) \\
&= -\int_{C_0\mathbf{R}^m} f(\omega)\ell\left(\int_0^{\cdot} \alpha_s \, dB_s \right) dP(\omega)
\end{aligned}$$

as required. (The last equality being obvious in the (most relevant) case when $\ell(v) = \lambda(v_{t_0})$ some $\lambda \in (\mathbf{R}^m)^*$, some $0 \leqslant t_0 \leqslant T$, in which case $\dot{k}_s = \chi_{[0,t_0]}(s)\lambda$.) $\square$

9. Differential geometry of the space $\mathcal{H}^2$ of two-vectors

In this section we will give a bundle structure to the Bismut tangent bundle $\mathcal{H}$ and interpret the quantities Q and $\mathbb{R}$ which define $\mathcal{H}^2$ in terms of a natural connection on $\mathcal{H}$.

A. *The L^2 tangent bundle and its frame bundle.* Let $\pi : OM \to M$ be the orthonormal frame bundle of M. Our Banach manifold $C_{x_0}M$ has natural structural group $C_{id}([0,T]; O(n))$ with frame bundle identified with the space of paths $C_{\pi^{-1}(x_0)}([0,T]; OM)$ in the frame bundle OM of M, starting at any frame over x_0. Let

$$\tilde{\pi} : C_{\pi^{-1}(x_0)}(OM) \to C_{x_0}M$$

be the projection. Note that $C_{id}(O(n))$ has an orthogonal representation on $L^2([0,T]; \mathbf{R}^n)$, acting pointwise

$$C_{id}\big(O(n)\big) \xrightarrow{\rho} O\big(L^2\big([0,T];\mathbf{R}^n\big)\big),$$
$$\rho(\alpha)(f)(t) = \alpha(t)\big(f(t)\big).$$

For α, β in $C_{id}(O(n))$,

$$\begin{aligned}\big\|\rho(\alpha)-\rho(\beta)\big\|_{\mathbb{L}(L^2([0,T];\mathbf{R}^n);L^2([0,T];\mathbf{R}^n))} &= \sup_{\|f\|_{L_2}\leqslant 1}\sqrt{\int_0^T \big|\alpha(s)f(s)-\beta(s)f(s)\big|^2\,ds} \\ &\leqslant \sup_{\|f\|_{L_2}\leqslant 1}\sqrt{\int_0^T \big|f(s)\big|^2 \sup_{0\leqslant s\leqslant T}\big|\alpha(s)-\beta(s)\big|^2\,ds} \\ &\leqslant \sup_{0\leqslant s\leqslant T}\big|\alpha(s)-\beta(s)\big| = d(\alpha,\beta).\end{aligned}$$

Thus ρ is continuous into the uniform topology and we see it is even C^∞ with derivative map $T_\alpha\rho$ at α:

$$T_\alpha\rho : T_\alpha C_{id}O(n) \to TO\big(L^2\big([0,T];\mathbf{R}^n\big)\big) \subset \mathcal{L}\big(L^2\big([0,T];\mathbf{R}^n\big);L^2\big([0,T];\mathbf{R}^n\big)\big)$$

given by $T_\alpha\rho(V)(f)(t) = V(t)f(t)$.

From this we see that the L^2 tangent bundle $L^2TC_{x_0}M$ has the structure of a C^∞ bundle associated to $C_{\pi^{-1}(x_0)}(OM)$, whose elements u act as frames on it by

$$u : L^2\big([0,T];\mathbf{R}^n\big) \to L^2T_\sigma C_{x_0}M, \quad \sigma = \tilde{\pi}u,$$
$$u(f)_t = u_t\big(f(t)\big).$$

This construction determines $L^2TC_{x_0}M$ as a C^∞ bundle over $C_{x_0}M$. It tells us what its smooth sections (in the Fréchet sense) are. (For example see Remark 9.1 below.)

B. *The pointwise connection.* Let $\tilde{\nabla}$ denote the *pointwise connection* on $C_{x_0}M$, as described in greater generality by Eliasson [25]. It is defined on the bundle $L^2TC_{x_0}M \to C_{x_0}M$ by

$$(\tilde{\nabla}_V U)_t = \frac{D}{ds}U\big(\exp_{\sigma_\cdot}(sV_\cdot)\big)_t\Big|_{s=0} \tag{9.1}$$

where $\frac{D}{ds}$ and exp come from the Levi-Civita connection on TM. Thus

$$\begin{aligned}(\tilde{\nabla}_V U)_t &= X(\sigma_t)\frac{d}{ds}\Big(Y\big(\exp_{\sigma_t}(sV_t)\big)U\big(\exp_{\sigma_\cdot}(sV_\cdot)\big)_t\Big)\Big|_{s=0} \\ &= X(\sigma_t)d\big[\tilde{Y}(\cdot)U(\cdot)\big](V)_t,\end{aligned}$$

where the L^2-valued one-form $\tilde{Y} : L^2TC_{x_0}M \to L^2([0,T];\mathbf{R}^m)$ is the lift of Y, i.e.

$$\tilde{Y}_\sigma(V)(t) = Y_{\sigma(t)}\big(V(t)\big).$$

This says that the pointwise connection is the LW connection in the sense of [30], for the lift $\tilde{X}$ of X to $C_{x_0}M$.

This connection is torsion free and is metric for the L^2 metric.

Remark 9.1. The pointwise derivative $\tilde{\nabla}Y : TC_{x_0}M \times L^2TC_{x_0}M \to L^2([0,T];\mathbf{R}^m)$ is C^∞.

To see this let Υ be a locally defined C^∞ frame field for $L^2TC_{x_0}M$ giving a local trivialisation over an open subset U of $C_{x_0}M$

$$\Upsilon : U \times L^2\big([0,T];\mathbf{R}^m\big) \to L^2TC_{x_0}M.$$

Then

$$\big[\tilde{Y}_\sigma \Upsilon(\sigma)(f)\big]_t = Y_{\sigma_t}\big(\Upsilon(\sigma)_t f(t)\big).$$

Its derivative is

$$(\nabla_{v_t}\tilde{Y})\Upsilon(\sigma)_t f(t) + Y_{\sigma_t}\big(\tilde{\nabla}_v \Upsilon\big(f(t)\big)\big).$$

C. *The bundle structure of $\mathcal{H}$ and its damped Markovian connection.* Let $C^0_{x_0}M$ be a set of paths of full measure along each element of which the Levi-Civita parallel translation, $/\!/$, is defined and satisfies its basic composition properties. Then $\mathcal{H}_\sigma$ is defined for each $\sigma \in C^0_{x_0}M$ by formula (4.5) with an isometry $\mathcal{W}_\sigma : L^2T_\sigma C_{x_0}M \to \mathcal{H}_\sigma$, with inverse $\frac{\mathbb{D}}{d\cdot}$. Thus we get an induced smooth vector bundle structure on $\mathcal{H}^1$, over $C^0_{x_0}M$ by

$$\frac{\mathbb{D}}{ds} : \mathcal{H}^1 \leftrightarrows L^2TC_{x_0}M.$$

We can use this isomorphism to pull back the pointwise connection to get a metric connection $\mathbb{V}$ on $\mathcal{H}^1$. This is the damped Markovian connection defined in a different way by Cruzeiro and Fang in [15,16], Cruzeiro, Fang and Malliavin [17]. The basis for a covariant Sobolev calculus using it is given in [35]. In particular we have a closed covariant derivative operator $\mathbb{V}$ with domain, denoted by $\mathbb{D}^{2,1}\mathcal{H}^1$, in the space of L^2 sections of $\mathcal{H}^1$ mapping to the L^2 sections of $\mathcal{L}_2(\mathcal{H}^1;\mathcal{H}^1)$. In general we shall not distinguish between $C^0_{x_0}M$ and $C_{x_0}M$.

Since the inverse map to $\frac{\mathbb{D}}{d\cdot}$ is $\mathcal{W}$ it follows from Eq. (4.2) that this connection is the LW connection associated to $\overline{T\mathcal{I}}$ in the sense of [30]. With this in mind define

$$\begin{aligned} \mathbb{X} : C_{x_0}M \times H &\to \mathcal{H}^1, \\ \mathbb{X}(\sigma)(h) &= \overline{T\mathcal{I}}(h). \end{aligned} \tag{9.2}$$

As noted in [35] the adjoint of $\mathbb{X}$ is the H-valued H-one-form $\mathbb{Y}$ given by

$$\mathbb{Y}_\sigma(V) = \int_0^{\cdot} Y^*_{\sigma(r)} \frac{\mathbb{D}}{dr} V_r \, dr.$$

This is also a right inverse to $\mathbb{X}$. Suppose that u^1 and u^2 are in $\mathbb{D}^{2,1}\mathcal{H}$. For $j=1,2$, set $h^j(\sigma)=\mathbb{Y}_\sigma(u^j(\sigma))$. Then, by [35], $h^j\in\mathbb{D}^{2,1}(C_{x_0}M;H)$ and

$$\begin{aligned}\mathbb{V}_{u^1(\sigma)}u^2&=\mathbb{X}(\sigma)\,\bar{d}\big[\mathbb{Y}(u^2)\big]\big(u^1(\sigma)\big)\\&=\mathbb{X}(\sigma)\,\bar{d}h^2\big(\overline{T\mathcal{I}}\big(h^1(\sigma)\big)\big)=\mathbb{X}(\sigma)\big(\overline{\bar{d}\big(h^2\circ\mathcal{I}\big)}_\sigma\big(h^1(\sigma)\big)\big).\end{aligned}\tag{9.3}$$

We saw in Proposition 8.1 that for certain v^1 and v^2 the two-vector field $Q(v^1\wedge v^2)$ has a divergence. After the following lemma we can identify that divergence:

Lemma 9.2. *Suppose $h:C_0\mathbf{R}^m\to H$ is adapted. Then*

$$\overline{T\mathcal{I}(h)}=\overline{T\mathcal{I}}(\bar{h}).$$

Proof. Set $v_t=T\mathcal{I}_t(h)$. Then, since h is adapted we have as for Eq. (3.6)

$$Dv_t=\nabla_{v_t}X(\tilde{/\!/}_t\,d\beta_t)-\frac{1}{2}\operatorname{Ric}^\#(v_t)\,dt+X(x_t)\dot{h}_t\,dt.$$

Now take conditional expectations as usual to get the result. □

Theorem 9.3. *For any $\mathcal{F}^{x_0}_\star$ adapted vector fields $u^i\in L^p\Gamma\mathcal{H}^1$, $i=1,2$, some $p>2$,*

$$\operatorname{div}Q\big(u^1\wedge u^2\big)=\frac{1}{2}\mathbb{T}\big(u^1,u^2\big),\tag{9.4}$$

where $\mathbb{T}$ is the torsion of the damped Markovian connection $\mathbb{V}$.

Proof. As above set $h^j=\mathbb{Y}(u^j)$, $j=1,2$. Define the adapted H-vector fields $\tilde{h}^j$, $j=1,2$, on $C_0\mathbf{R}^m$ by $\tilde{h}^j=h^j\circ\mathcal{I}$. First assume that each u^j, and so h^j and $\tilde{h}^j$, belong to $\mathbb{D}^{p,1}$.

By the integration by parts formulae, as for the proof of (8.5) for two-vector fields in Section 8, and using the fact that $\dot{\tilde{h}}^j(\omega)_s\perp\ker X(x_s(\omega))$ a.s.,

$$\begin{aligned}\operatorname{div}\big(u^j\big)\circ\mathcal{I}&=\mathbf{E}\big\{\operatorname{div}(\tilde{h}^j)\big|\mathcal{F}^{x_0}\big\}=-\mathbf{E}\left\{\int_0^T\langle\dot{h}_s,dB_s\rangle\Big|\mathcal{F}_{x_0}\right\}\\&=-\int_0^T\big\langle\dot{\tilde{h}}^j_s,X(x_s)\,dB_s\big\rangle=\operatorname{div}\big(\tilde{h}^j\big).\end{aligned}$$

In particular $\operatorname{div}(\tilde{h}^j)$ is $\mathcal{F}^{x_0}$-measurable. Consequently, from Proposition 8.1 and formula (8.3),

$$\begin{aligned}2\operatorname{div}Q\big(u^1\wedge u^2\big)&=2\overline{T\mathcal{I}\big(\operatorname{div}\big(\tilde{h}^1\wedge\tilde{h}^2\big)\big)}-2\operatorname{div}\big(u^1\wedge u^2\big)\\&=\overline{T\mathcal{I}\big(-\tilde{h}^1\operatorname{div}\big(\tilde{h}^2\big)+\tilde{h}^2\operatorname{div}\big(\tilde{h}^1\big)+\big[\tilde{h}^1,\tilde{h}^2\big]\big)}\\&\quad-\big(\operatorname{div}u^1\big)u^2+u^1\operatorname{div}\big(u^2\big)-\big[u^1,u^2\big]\\&=\overline{T\mathcal{I}\big(\big[\tilde{h}^1,\tilde{h}^2\big]\big)}-\big[u^1,u^2\big].\end{aligned}$$

Also from (9.3)

$$\begin{aligned}
\big[u^1, u^2\big](\sigma) &= \mathbb{X}(\sigma)\big(\overline{(\bar{d}\tilde{h}^2)}_\sigma\big(h^1(\sigma)\big) - \overline{(\bar{d}\tilde{h}^1)}_\sigma\big(h^2(\sigma)\big)\big) - \mathbb{T}\big(u^1, u^2\big)(\sigma) \\
&= \mathbb{X}(\sigma)\big(\overline{\big[\tilde{h}^1, \tilde{h}^2\big]}\big)_\sigma - \mathbb{T}\big(u^1, u^2\big)(\sigma) \\
&= \overline{T\mathcal{I}}_\sigma\big(\overline{\big[\tilde{h}^1, \tilde{h}^2\big]}_\sigma\big) - \mathbb{T}\big(u^1, u^2\big)(\sigma)
\end{aligned}$$

giving

$$2\operatorname{div} Q\big(u^1 \wedge u^2\big)(\sigma) = \overline{T\mathcal{I}\big(\big[\tilde{h}^1, \tilde{h}^2\big]\big)}_\sigma - \overline{T\mathcal{I}}_\sigma\big(\overline{\big[\tilde{h}^1, \tilde{h}^2\big]}_\sigma\big) + \mathbb{T}\big(u^1, u^2\big)(\sigma).$$

For adapted vector fields the first two terms cancel by the previous lemma, so we have (9.4) for adapted $\mathbb{D}^{p,1}$ vector fields.

If u^1, u^2 are adapted but not in $\mathbb{D}^{p,1}$ we can choose, cf. Lemma 9.4, sequences of adapted processes $\{u_n^j\}_{n=1}^\infty$, $j = 1, 2$, in $\mathbb{D}^{p,1}\mathcal{H}$, converging to u^1, u^2 in L^p. Then as $n \to \infty$,

$$\mathbb{T}\big(u_n^1, u_n^2\big) \to \mathbb{T}\big(u^1, u^2\big)$$

in $L^1 T C_{x_0} M$, by the formula

$$\mathbb{T}\big(V^1, V^2\big) = \tilde{X}\big((\nabla_{V^2}\tilde{Y})V^1 - (\nabla_{V^1}\tilde{Y})V^2\big)$$

given in Appendix B. On the other hand, for any C^∞ cylindrical 1-form ϕ,

$$\int \phi\big(\mathbb{T}\big(u_n^1, u_n^2\big)\big) = -2\int d\phi\big(Q\big(u_n^1 \wedge u_n^2\big)\big) \to -2\int d\phi\big(Q\big(u^1 \wedge u^2\big)\big).$$

Thus for all adapted L^p vector fields u^i, we have

$$\operatorname{div} Q\big(u^1 \wedge u^2\big) = \frac{1}{2}\mathbb{T}\big(u^1, u^2\big). \qquad \square$$

Lemma 9.4. *If u is an $\mathcal{F}_\star^{x_0}$-adapted H-vector field in $L^p \Gamma \mathcal{H}^1$ for some $p > 1$, there is a sequence $u_n \in \mathbb{D}^{p,1}\mathcal{H}^1$ of $\mathcal{F}_\star^{x_0}$ adapted H-vector fields such that u_n converges to u in L^p.*

Proof. Set $\tilde{h} = \mathbb{Y}(\frac{d}{d\cdot}u) \circ \mathcal{I} \in L^p(C_0\mathbf{R}^m; L^2([0, T]; \mathbf{R}^m))$. As finite chaos expansions are dense in L^p, let $\{\tilde{h}_n\}$ be a sequence of functions with finite chaos expansion converging to $\tilde{h}$ in $L^p(C_0\mathbf{R}^m; L^2([0, T]; \mathbf{R}^m))$. Define $v_n : C_{x_0} \to L^2([0, T]; \mathbf{R}^m)$ by

$$(v_n \circ \mathcal{I})_t = \mathbf{E}\big\{\tilde{h}_n \big| \mathcal{F}_t^{x_0}\big\}.$$

Then v_n belongs to $\mathbb{D}^{p,1}$, see [35]. Set $u_n = \mathbb{X}(\int_0^\cdot (v_n)_s\, ds)$ then u_n converges in L^p to u. $\square$

Remark 9.5.

(1) It is noted in Cruzeiro and Fang [16] that the divergence of $\mathbb{T}(v^1, v^2)$ vanishes for a class of adapted H-vector fields v^1 and v^2.

(2) The conclusion of the theorem does not hold for general smooth nonadapted vector fields. In fact for a smooth, cylindrical $f : C_{x_0}M \to \mathbf{R}$ we have $\mathbb{T}(f\bar{v}^1, \bar{v}^2) = f\mathbb{T}(\bar{v}^1, \bar{v}^2)$. But

$$\operatorname{div} Q\big((f\bar{v}^1) \wedge \bar{v}^2\big) = \operatorname{div}\big(f Q(\bar{v}^1 \wedge \bar{v}^2)\big) = f \operatorname{div}\big(Q(\bar{v}^1 \wedge \bar{v}^2)\big) + \iota_{\nabla f} Q(\bar{v}^1 \wedge \bar{v}^2).$$

Though we state the following for Brownian motion measures and the damped Markovian connections note that it applies in considerable generality, for example for any metric connection on a finite dimensional Riemannian manifold with smooth measure. In it we consider the closed covariant derivative operator

$$\mathbb{V} : \mathbb{D}^{2,1} \subset L^2\Gamma\mathcal{H}^1 \to L^2\Gamma\mathcal{L}_2(\mathcal{H}^1; \mathcal{H}^1)$$

with L^2-adjoint $\mathbb{V}^* : L^2\Gamma\mathcal{L}(\mathcal{H}^1; \mathcal{H}^1) \to L^2\Gamma\mathcal{H}^1$.

Proposition 9.6. *Let $U, V \in L^\infty\Gamma\mathcal{H}^1$. Suppose $U \in \mathbb{D}^{2,1}$ and $V \in$ Dom(div). Then $\sigma \mapsto U(\sigma) \otimes V(\sigma)$ as an element, $U \otimes V$, of $L^2\Gamma(\mathcal{H}^1 \otimes \mathcal{H}^1)$ is in* Dom($\mathbb{V}^*$) *and*

$$\mathbb{V}^*(U \otimes V)(\sigma) = -(\operatorname{div} V)(\sigma)U(\sigma) - \mathbb{V}_{V(\sigma)}U. \tag{9.5}$$

In particular this holds if U and V are both essentially bounded and in $\mathbb{D}^{2,1}$ in which case:

$$\mathbb{V}^*(U \wedge V) = \operatorname{div}(U \wedge V) + \frac{1}{2}\mathbb{T}(U, V). \tag{9.6}$$

Proof. Let $Z \in \mathbb{D}^{2,1}\mathcal{H}^1$. By (6.1) and (6.2),

$$\begin{aligned}
&\int_{C_{x_0}M} \big\langle (\mathbb{V}Z)_\sigma, U \otimes V(\sigma)\big\rangle_{\mathcal{H}^1_\sigma \otimes \mathcal{H}^1_\sigma} \, d\mu_{x_0}(\sigma) \\
&\quad = \int_{C_{x_0}M} \big\langle (\mathbb{V}Z)_\sigma, U \otimes V(\sigma)\big\rangle_{\mathcal{L}_2(\mathcal{H}^1;\mathcal{H}^1)} \, d\mu_{x_0}(\sigma) \\
&\quad = \int_{C_{x_0}M} \sum_{i=1}^\infty \big\langle (\mathbb{V}_{e_i}Z)_\sigma, U(\sigma)\big\langle V(\sigma), e_i\big\rangle\big\rangle_{\mathcal{H}^1_\sigma} \, d\mu_{x_0}(\sigma) \\
&\quad = \int_{C_{x_0}M} \big\langle \mathbb{V}_{V(\sigma)}Z, U(\sigma)\big\rangle_{\mathcal{H}^1_\sigma} \, d\mu_{x_0}(\sigma) \\
&\quad = \int_{C_{x_0}M} d\langle Z, U\rangle_{\mathcal{H}^1}\big(V(\sigma)\big) \, d\mu_{x_0}(\sigma) - \int_{C_{x_0}M} \langle Z, \mathbb{V}_{V(\sigma)}U\rangle_{\mathcal{H}^1_\sigma} d\mu_{x_0}(\sigma),
\end{aligned}$$

since $\mathbb{V}$ is a metric connection. This proves the first part.

For the second part first note from [35] that H-vector fields which are in $\mathbb{D}^{2,1}$ are in Dom(div). Then plug $U \wedge V = \frac{1}{2}\{U \otimes V - V \otimes U\}$ into Eq. (9.5) and use formula (8.3) to see

$$\mathbb{V}^*(U \wedge V) = \frac{1}{2}\big\{-(\operatorname{div} V)U + (\operatorname{div} U)V - \mathbb{V}_V U + \mathbb{V}_U V\big\}$$
$$= \operatorname{div}(U \wedge V) + \frac{1}{2}\mathbb{T}(U, V). \qquad \square$$

By formula (9.4) this immediately gives

Corollary 9.7. *For U, V as in Proposition* 9.6

$$\mathbb{V}^*(U \wedge V) = \operatorname{div}(I + Q)(U \wedge V) \tag{9.7}$$

provided U, V are non-anticipating. In particular for h^1, h^2 in $L_0^{2,1}(\mathbf{R}^m)$ non-random

$$\operatorname{div}\big(\overline{\textstyle\bigwedge^2 T\mathcal{I}}\big(h^1 \wedge h^2\big)\big) = \nabla^*\big(\overline{T\mathcal{I}}\big(h^1\big) \wedge \overline{T\mathcal{I}}\big(h^2\big)\big). \tag{9.8}$$

Note that for Z as above, if $f : C_{x_0}M \to \mathbf{R}$ is smooth and cylindrical then

$$\int_{C_{x_0}M} \langle \mathbb{V}Z, fU \wedge V\rangle_{\mathcal{H}^1\otimes\mathcal{H}^1}\, d\mu_{x_0}$$
$$= \int_{C_{x_0}M} \big\langle \mathbb{V}(fZ) - Z \otimes \nabla f, U \wedge V\big\rangle_{\mathcal{H}^1\otimes\mathcal{H}^1}\, d\mu_{x_0}$$
$$= \int_{C_{x_0}M} \left\{\big\langle Z, f\,\mathbb{V}^*(U \wedge V)\big\rangle - \frac{1}{2}\langle Z, U\rangle\, df(V) + \frac{1}{2}\langle Z, V\rangle\, df(U)\right\} d\mu_{x_0}.$$

So

$$\mathbb{V}^*[fU \wedge V] = f\,\mathbb{V}^*(U \wedge V) - \frac{1}{2}\big\{U\, df(V) - V\, df(U)\big\}$$
$$= f\,\mathbb{V}^*(U \wedge V) + \iota_{\nabla f}(U \wedge V)$$

whereas

$$\operatorname{div}(I + Q)(fU \wedge V) = f\operatorname{div}(I + Q)(U \wedge V) + \iota_{\nabla f}(U \wedge V) + \iota_{df}\, Q(U \wedge V).$$

Thus the formula is not true, if ‘non-anticipating’ is dropped.

D. *The curvature operator.* The curvature operator $\mathbb{R}$ of the damped Markovian connection $\mathbb{V}$ on $\Gamma\mathcal{H}^1$ is conjugate to the curvature operator

$$\tilde{\mathcal{R}} : \textstyle\bigwedge^2 TC_{x_0}M \to \mathcal{L}_{skew}\big(L^2TC_{x_0}M; L^2TC_{x_0}M\big)$$

of the pointwise connection on the L^2 tangent bundle via the map $\frac{\mathbb{D}}{dt}$. In fact

$$\mathbb{R} : \textstyle\bigwedge^2 T_\sigma C_{x_0}M \to \mathcal{L}_{skew}\big(\mathcal{H}^1_\sigma; \mathcal{H}^1_\sigma\big)$$

is given by

$$\big(\mathbb{R}(U)h\big)_t = \mathcal{W}_t\left(\tilde{\mathcal{R}}_\sigma\big(U(\sigma)\big)\left(\frac{\mathbb{D}}{d\cdot}h_\cdot\right)\right),$$

that is

$$\big(\mathbb{R}(U)h\big)_t = W_t\int_0^t W_s^{-1}\mathcal{R}_{\sigma_s}(U_{s,s})\left(\frac{\mathbb{D}}{d\cdot}h_s\right)ds. \tag{9.9}$$

We shall show that this agrees with the definition given in Eq. (4.17).

Our convention that $(a\otimes b)(u)=\langle b,u\rangle a$ makes clear the correspondence between the curvature operator $\mathcal{R}$ of M considered as a map $\mathcal{R}:\bigwedge^2 TM\to\mathcal{L}(TM;TM)$ and it considered as a map $\mathcal{R}:\bigwedge^2 TM\to\bigwedge^2 TM$. Note also that for a linear map T

$$\big[(T\otimes\mathbf{1})(a\otimes b)\big](u) = T\big((a\otimes b)(u)\big).$$

Then

$$\begin{aligned}\mathbb{R}(U)(h)_t &= W_t\int_0^t W_r^{-1}\mathcal{R}(U_{rr})\left(\frac{\mathbb{D}}{dr}h_r\right)dr = W_t\int_0^t\big[\big(W_r^{-1}\otimes\mathbf{1}\big)\mathcal{R}(U_{rr})\big]\left(\frac{\mathbb{D}}{dr}h_r\right)dr\\ &= \int_0^t\big[\big(W_t(W_r)^{-1}\otimes\mathbf{1}\big)\mathcal{R}(U_{rr})\big]\left(\frac{\mathbb{D}}{dr}h_r\right)dr\\ &= \int_0^T\chi_{[0,t)}(r)(W_t\otimes W_r){\textstyle\bigwedge^2}\big(W_r^{-1}\big)\mathcal{R}(U_{rr})\left(\frac{\mathbb{D}}{dr}h_r\right)dr.\end{aligned}$$

Proposition 9.8. *As a linear map from $\bigwedge^2 T_\sigma C_{x_0}M$ to $\bigwedge^2 TC_{x_0}M$, the curvature operator of the damped Markovian connection on $\mathcal{H}^1$ is given by*

$$\mathbb{R}(U)_{s,t} = (W_s\otimes W_t)\int_0^t {\textstyle\bigwedge^2}(W_r)^{-1}\mathcal{R}(U_{rr})\,dr,\quad t<s. \tag{9.10}$$

Proof. Since $\mathbb{R}(U)$ is regular, its integral representation is

$$\mathbb{R}(U)(h_\cdot)_t = \int_0^T\left(\mathbf{1}\otimes\frac{\mathbb{D}}{dr}\right)\mathbb{R}(U)_{t,r}\left(\frac{\mathbb{D}h_r}{dr}\right)dr.$$

Compare this with the integral representation above the proposition to see the result. □

E. *The domain of* $\bar{d}^{1*}$. An important result for functions on $C_0\mathbf{R}^m$ was that the domain of the divergence acting on H-vector fields contains $\mathbb{D}^{2,1}(C_0\mathbf{R}^m; H)$, in particular H-vector fields which are in $\mathbb{D}^{2,1}$ are Skorohod integrable [51]. For $C_{x_0}M$ the analogous result was proved in [35] using the damped Markovian connection. We have not yet given a "bundle structure" or connection to $\mathcal{H}^2$ or its dual, but $\bigwedge^2 L^2TC_{x_0}M$ is a smooth bundle and inherits a connection from the pointwise connection. This will be the LW connection for $\bigwedge^2 \tilde{X}$. As discussed, in general, in [35] a section Z of $\bigwedge^2 L^2TC_{x_0}M$ is in $\mathbb{D}^{2,1}\bigwedge^2 L^2TC_{x_0}M$ if $\bigwedge^2 \tilde{Y}(Z)$ is in $\mathbb{D}^{2,1}(C_{x_0}M; \bigwedge^2 L^2([0,T];\mathbf{R}^m))$. Where defined, the map

$$(\mathbf{1}+Q)\bigwedge^2 \mathcal{W} : \bigwedge^2 L^2TC_{x_0}M \to \mathcal{H}^2$$

is an isometry and it would be natural to use this to give a connection on $\mathcal{H}^2$. In this sense the following shows that the results mentioned above extend to our H-two-forms (or equivalently for the divergence operator on H-two-vectors). It is stated in terms of the weak Sobolev class $W^{2,1}$ for, possibly, greater generality.

Theorem 9.9. 1. *Let* $\phi \in L^2\Gamma\mathcal{H}^2$. *If*

$$\phi \circ (\mathbf{1}+Q) \circ \bigwedge^2 \mathcal{W} \in W^{2,1}\Gamma \bigwedge^2 \big(L^2TC_{x_0}M\big)^*$$

then $\phi \in \mathrm{Dom}(\bar{d}^{1*})$.

2. *More generally* $\phi \in \mathrm{Dom}(\bar{d}^{1*})$ *if the conditional expectation of its pull back by the Itô map*

$$\mathbf{E}\big\{\mathcal{I}^*(\phi)\big|\mathcal{F}^{x_0}\big\} : C_0\mathbf{R}^m \to \big(\bigwedge^2 H\big)^*$$

is in the domain of $\bar{d}^{1*}$ *on* $C_0\mathbf{R}^m$. *If so, for almost all* $\sigma \in C_{x_0}M$ *the H-vector field* $\mathrm{div}\,\phi^\#$ *is given by*

$$\mathrm{div}\big(\phi^\#\big) = \overline{T\mathcal{I}\big(\mathrm{div}\big(\mathbf{E}\big\{\mathcal{I}^*(\phi)\big|\mathcal{F}^{x_0}\big\}\big)^\#\big)}.$$

Proof. Set

$$g(\sigma) = \phi \circ (\mathbf{1}+Q) \circ \bigwedge^2 \mathcal{W} \circ \bigwedge^2 \tilde{X}(\sigma) \circ \bigwedge^2 \left(\frac{d}{d\cdot}\right)$$

for $\sigma \in C_{x_0}M$. Then our first condition implies that $g \in W^{2,1}(C_{x_0}M; \bigwedge^2 H)$. Note that $g = \phi \circ (\mathbf{1}+Q) \circ \bigwedge^2 \mathcal{W} \circ \bigwedge^2 \mathbb{X}$ and so

$$g \circ \mathcal{I} = \mathbf{E}\big\{\mathcal{I}^*(\phi)\big|\mathcal{F}^{x_0}\big\}.$$

By [35] $g \circ \mathcal{I} \in \mathbb{D}^{2,1}$ on $C_0\mathbf{R}^m$. By [69] this implies that as an H-two-form $g \circ \mathcal{I}$ is in the domain of d^{1*}. Now let $\psi \in \mathrm{Dom}(d^1_{\mathcal{H}})$, cylindrical one-form on $C_{x_0}M$. Then we have

$$\int\limits_{C_{x_0}M} \langle d^1_{\mathcal{H}}\psi, \phi\rangle_{\mathcal{H}^{2*}} = \int\limits_{C_{x_0}} \big\langle d^1_{\mathcal{H}}\psi\big(\overline{\bigwedge^2 T\mathcal{I}}(-)\big), \phi\big(\overline{\bigwedge^2 T\mathcal{I}}(-)\big)\big\rangle_{(\bigwedge^2 H)^*}$$

$$= \int_{C_0\mathbf{R}^m} \left\langle d^1_{\mathcal{H}}\psi \bigwedge^2 T\mathcal{I}(-), \mathbf{E}\left\{\mathcal{I}^*(\phi)\middle|\mathcal{F}^{x_0}\right\}\right\rangle_{\bigwedge^2 H^*}$$

$$= \int_{C_0\mathbf{R}^m} \left\langle \mathcal{I}^*\left(d^1_{\mathcal{H}}\psi\right), \mathbf{E}\left\{\mathcal{I}^*(\phi)\middle|\mathcal{F}^{x_0}\right\}\right\rangle_{\bigwedge^2 H^*}$$

$$= \int_{C_0\mathbf{R}^m} \left\langle \bar{d}^1\mathcal{I}^*(\psi), \mathbf{E}\left\{\mathcal{I}^*(\phi)\middle|\mathcal{F}^{x_0}\right\}\right\rangle_{\bigwedge^2 H^*}$$

$$= \int_{C_0\mathbf{R}^m} \left\langle I^*(\psi), \left(\bar{d}^1\right)^*\mathbf{E}\left\{\mathcal{I}^*(\phi)\middle|\mathcal{F}^{x_0}\right\}\right\rangle_{\bigwedge^2 H^*}.$$

From this the results follow. □

Corollary 9.10. *Every C^1 cylindrical 2-form on $C_{x_0}M$ is in the domain of $\bar{d}^{1*}$.*

Proof. Let $M^{(k)} = M \times M \times \cdots \times M$ be the Cartesian product of k copies of M and for $0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant T$ define $\rho_{\underline{t}} : C_{x_0}M \to M^{(k)}$ by $\rho_{\underline{t}}(\sigma) = (\sigma(t_1), \ldots, \sigma(t_k))$. Suppose $\phi = \rho_{\underline{t}}^*(\theta)$ for θ a C^1 two-form on $M^{(k)}$. Then

$$\begin{aligned}\mathbf{E}\left\{\mathcal{I}^*(\phi)\middle|\mathcal{F}^{x_0}\right\} &= \phi_{\mathcal{I}(\cdot)} \circ (\mathbf{1} + Q_{\mathcal{I}(\cdot)}) \circ \bigwedge^2 \mathbb{X}\left(\mathcal{I}(\cdot)\right)\\ &= \theta \circ \bigwedge^2 X^{(k)}\left(\mathcal{I}(\rho_{\underline{t}(\cdot)})\right) \circ \bigwedge^2 Y^{(k)} \bigwedge^2 T\rho_{\underline{t}} \circ (\mathbf{1} + Q_{\mathcal{I}(\cdot)}) \circ \bigwedge^2 \mathbb{X}\left(\mathcal{I}(\cdot)\right)\end{aligned}$$

where $X^{(k)}(z_1, \ldots, z_k) = \bigoplus_{j=1}^k X(z_j) : \bigoplus^k \mathbf{R}^m \to T_z M^{(k)}$ for $z = (z_1, \ldots, z_k) \in M^{(k)}$. Now, from the differentiability of θ and X it is clear that $\theta \circ \bigwedge^2 X^{(k)}(\mathcal{I}(\rho_{\underline{t}(\cdot)}))$ is in $\mathbb{D}^{p,1}$ for all $1 \leqslant p < \infty$, while it follows from standard approximation arguments that so is $\bigwedge^2 Y^{(k)} \bigwedge^2 T\rho_{\underline{t}} \circ (\mathbf{1} + Q_{\mathcal{I}(\cdot)}) \circ \bigwedge^2 \mathbb{X}(\mathcal{I}(\cdot))$, for example as in [2]. Thus we can apply the theorem as required. □

Appendix A. Conventions

In the past we have used different conventions on the exterior product of a differential form, inner product of two antisymmetric tensor vectors, and the interior product of a vector with another. Here we were driven by the insistence that exterior product spaces are subspaces of the corresponding tensor products. To make these differences more transparent and easier for the reader to compare to their own conventions, we list in this section the conventions we use. It is only necessary to state them for uncompleted tensor products.

A. Let E, F be a real linear spaces. Any multilinear $\psi : E \times E \times \cdots \times E \to F$ determines a linear map $\tilde{\psi} : E \otimes_0 E \otimes_0 \cdots \otimes_0 E \to F$ with

$$\tilde{\psi}(u_1 \otimes \cdots \otimes u_q) = \psi(u_1, \ldots, u_q).$$

Denote by $\bigwedge_0^q E$ the subspace of anti-symmetric tensors and use the convention

$$u_1 \wedge \cdots \wedge u_q = \frac{1}{q!} \sum_{\pi} (-1)^{\pi} u_{\pi(1)} \otimes \cdots \otimes u_{\pi(q)} \tag{A.1}$$

where the summation is over all permutations π of $\{1,2,\dots,q\}$ and $(-1)^\pi$ is the sign of the permutation. This convention ensures that if ψ is anti-symmetric then

$$\tilde{\psi}(u_1 \wedge \cdots \wedge u_q) = \psi(u_1, \dots, u_q).$$

An inner product $\langle -,- \rangle$ on E determines one on the tensor products

$$\langle u_1 \otimes \cdots \otimes u_q, v_1 \otimes \cdots \otimes v_q \rangle = \prod_{i=1}^{q} \langle u_i, v_i \rangle, \tag{A.2}$$

any $u_i, v_i \in E$. In turn this determines one on the exterior powers by restriction, giving

$$\langle u_1 \wedge \cdots \wedge u_q, v_1 \wedge \cdots \wedge v_q \rangle = \frac{1}{q!} \det \begin{pmatrix} \langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \dots & \langle u_1, v_q \rangle \\ \dots & \dots & \dots & \dots \\ \langle u_q, v_1 \rangle & \langle u_q, v_2 \rangle & \dots & \langle u_q, v_q \rangle \end{pmatrix}. \tag{A.3}$$

Now suppose there is a pairing $\langle\langle -,- \rangle\rangle : E' \times E \to \mathbf{R}$ between E and a linear space E'. We are thinking of the cases $E = E'$ with inner product pairing or E' being the dual space of E with respect to some topology, with $\langle\langle l, e \rangle\rangle = l(e)$. Then if $l \in E'$, there is the standard interior product, or annihilation operator ι_l,

$$\iota_l(u_1 \otimes \cdots \otimes u_q) = \langle\langle l, u_1 \rangle\rangle (u_2 \otimes \cdots \otimes u_q). \tag{A.4}$$

This gives

$$\iota_l(u_1 \wedge \cdots \wedge u_q) = \frac{1}{q} \sum_{j=1}^{q} (-1)^{j+1} \langle\langle l, u_j \rangle\rangle u_1 \wedge \cdots \wedge \hat{u}_j \wedge \cdots \wedge u_q \tag{A.5}$$

where $\hat{u}$ means the omission of the vector u. Note that:

(i) If $E = E'$ with inner product pairing then for each $v \in E$ the operator $\iota_v : \bigwedge_0^q E \to \bigwedge_0^{q-1} E$ is adjoint to the map determined by $u_1 \wedge \cdots \wedge u_{q-1} \to v \wedge u_1 \wedge \cdots \wedge u_{q-1}$.

(ii) The interior product is now not a skew-derivation, cf. [50, p. 65]. For example if $q = 2$ we have

$$\iota_l(u_1 \wedge u_2) = \frac{1}{2} \big\{ \langle\langle l, u_1 \rangle\rangle u_2 - \langle\langle l, u_2 \rangle\rangle u_1 \big\}.$$

Keeping the duality between the interior product and the "creation operator" $v \wedge -$, for ψ as above and $v \in E$ define:

$$\iota_v \psi : \mathsf{X}^{(q-1)} E \to \mathbf{R}$$

by

$$\iota_v(\psi)(u_1, \dots, u_{q-1}) = \psi(v, u_1, \dots, u_{q-1}),$$

so that if ψ is skew-symmetric we have

$$\iota_v(\psi)(u_1 \wedge \cdots \wedge u_{q-1}) = \psi(v \wedge u_1 \wedge \cdots \wedge u_{q-1}).$$

If ϕ_1 and ϕ_2 are in a dual space to E then $\phi_1 \wedge \phi_2$ is defined on $\bigwedge_0^2 E$ by

$$\phi_1 \wedge \phi_2(u_1 \wedge u_2) = \frac{1}{2}\big[\phi_1(u_1)\phi_2(u_2) - \phi_2(u_1)\phi_1(u_2)\big].$$

This is in agreement with $\iota_v(\phi_1 \wedge \phi_2) := \frac{1}{2}(\phi_1(v)\phi_2 - \phi_2(v)\phi_1)$.

B. More generally if $S : E_1 \to E_2$ and $T : F_1 \to F_2$ are two linear maps of Banach spaces, there is the induced linear map

$$S \otimes T : E_1 \otimes_0 F_1 \to E_2 \otimes_0 F_2.$$

If $E_1 = F_1$ and $E_2 = F_2$ set $S \wedge T = \frac{1}{2}(S \otimes T + T \otimes S)$ so $S \otimes S$ agrees with $S \wedge S$ as a linear operator on $\bigwedge^2 E_1$. This reduces to the previous definitions when $E_2 = F_2 = \mathbf{R}$ after identifying $\mathbf{R} \otimes \mathbf{R}$ with $\mathbf{R}$.

C. Consider now the tangent bundle TM of a smooth manifold M. The exterior differentiation $d : \bigwedge^q TM \to \bigwedge^{q+1} TM$ is defined by

$$\begin{aligned} &d\phi\big(V^1 \wedge \cdots \wedge V^{q+1}\big) \\ &\quad = \frac{1}{(q+1)} \sum_{i=1}^{q+1} (-1)^{i+1} L_{V^i}\big[\phi\big(V^1 \wedge \cdots \wedge \widehat{V^i} \wedge \cdots \wedge V^{q+1}\big)\big] \\ &\qquad + \frac{1}{(q+1)} \sum_{1 \leqslant i < j \leqslant q+1} (-1)^{i+j} \phi\big(\big[V^i, V^j\big] \wedge V^1 \wedge \cdots \wedge \widehat{V^i} \wedge \cdots \wedge \widehat{V^j} \wedge \cdots \wedge V^{q+1}\big) \end{aligned} \tag{A.6}$$

where L_{V^i} denotes Lie differentiation in the direction of v^i. This differs from the convention used in our previous research paper, e.g. [30,31,33] where we did not add any constants before d and d^*. This lead to a change in the divergence of q-vector fields by a factor of q

$$\mathrm{div}_{old}(V) = q \, \mathrm{div}_{new}(V). \tag{A.7}$$

By our conventions if f is a function on M,

$$\langle df \wedge \phi, \psi\rangle = \langle \phi, \iota_{df}\psi\rangle, \tag{A.8}$$

$$d(f\phi) = df \wedge \phi + f \, d\phi, \tag{A.9}$$

$$\mathrm{div}(fV) = f(\mathrm{div}\, V) + \iota_V(df). \tag{A.10}$$

Appendix B. Brackets of vector fields, torsion, and $d\phi(v^1 \wedge v^2)$

Lie brackets of H-vector fields have been discussed in many places, e.g. [19,21,55], for completeness, and definitiveness, we give a definition and some properties here. The torsion of the damped Markovian connection is also described, for explicit formulae see [16]. We refer to [35] for the Sobolev calculus of sections of $\mathcal{H}$, related bundles, and smooth bundles such as $L^2TC_{x_0}M$. The latter will always be taken here with its pointwise connection.

Proposition B.1. *The inclusion map of $\mathcal{H}$ into $L^2TC_{x_0}M$ is in $\mathbb{D}^{p,1}$ for $1 \leqslant p < \infty$ as a section of $\mathcal{L}_2(\mathcal{H}; L^2TC_{x_0}M)$ and any H-vector field V in $D^{p,1}\mathcal{H}$, or $\mathbb{W}^{p,1}\mathcal{H}$, is a $\mathbb{D}^{p,1}$, or $\mathbb{W}^{p,1}$, section of $L^2TC_{x_0}M$. Moreover for such V the pointwise (weak) covariant derivative $\tilde{\nabla}_- V$ is an L^p section of $\mathcal{L}(\mathcal{H}; TC_{x_0}M)$.*

Proof. For the first assertion it suffices to show that the map

$$\Theta : C_{x_0}M \to \mathcal{L}_2\big(H; L^2\big([0,T]; \mathbf{R}^m\big)\big)$$

given by

$$\Theta(\sigma)(h) = \tilde{Y}_\sigma \overline{T\mathcal{I}}_\sigma(h)$$

is in $\mathbb{D}^{p,1}$. However $\Theta(\sigma)(h)_t = Y_{\sigma(t)} W_t \int_0^t W_s^{-1} X(\sigma(s))(\dot{h}_s)\, ds$ and so the result holds from standard arguments, as in [2]. For the claim about sections we can apply the corresponding arguments to $\sigma \mapsto \Theta(\sigma)(U(\sigma))$ for $U \in \mathbb{D}^{p,1}(C_{x_0}M; H)$, or in $\mathbb{W}^{p,1}(C_{x_0}M; H)$; in the latter case it is only necessary to consider the composition with $\mathcal{I}$, see Theorem 5.1. In particular the final assertion comes from standard results giving the continuity in t of the derivative of $(\Theta \circ \mathcal{I})(U \circ \mathcal{I})_t : C_0\mathbf{R}^m \to \mathbf{R}^m$, e.g. as [64, p. 106]. Alternatively the derivative can be calculated explicitly as in [2]. □

Definition B.2. If V^1 and V^2 are in $\mathbb{W}^{p,1}\mathcal{H}$ define their Lie bracket by

$$\big[V^1, V^2\big] = \tilde{\nabla}_{V^1} V^2 - \tilde{\nabla}_{V^2} V^1,$$

where $\tilde{\nabla}$ is the pointwise connection defined by formula (9.1).

By Proposition B.1, $[V^1, V^2]$ is then a measurable vector field, i.e. section of $TC_{x_0}M$. Since the pointwise connection restricts to a torsion free connection on $TC_{x_0}M$ this definition agrees with the usual one. Moreover if $f : C_{x_0}M \to \mathbf{R}$ is smooth and cylindrical we have

$$\bar{d}\big(\bar{d}f\big(V^2\big)\big)V^1 = \tilde{\nabla}_{V^1}(\bar{d}f)V^2 + \bar{d}f\big(\tilde{\nabla}_{V^1}V^2\big)$$

so that

$$\bar{d}\big(\bar{d}f\big(V^2\big)\big)V^1 - \bar{d}\big(\bar{d}f\big(V^1\big)\big)V^2 = \bar{d}f\big(\big[V^1, V^2\big]\big)$$

as usual. The torsion $\mathbb{T}(V^1, V^2)$ is defined as a measurable vector field by

$$\mathbb{T}\big(V^1, V^2\big) = \boldsymbol{\nabla}_{V^1} V^2 - \boldsymbol{\nabla}_{V^2} V^1 - \big[V^1, V^2\big].$$

To see the torsion as an "H-tensor field" use the LW characterisation of the pointwise connection to observe first that

$$\begin{aligned}\mathbb{T}(V^1, V^2) &= \mathbb{V}_{V^1} V^2 - \mathbb{V}_{V^2} V^1 - \tilde{\nabla}_{V^1} V^2 + \tilde{\nabla}_{V^2} V^1 \\ &= \mathbb{V}_{V^1} V^2 - \tilde{X}\bar{d}(\tilde{Y} V^2) V^1 - \mathbb{V}_{V^2} V^1 + \tilde{X}\bar{d}(\tilde{Y} V^1) V^2.\end{aligned}$$

Now consider the restriction of $\tilde{Y}$ to $\mathcal{H}$ as a section of $\mathcal{L}_2(\mathcal{H}; L^2([0,T]; \mathbf{R}^m))$. As above it lies in $\mathbb{D}^{p,1}$, $1 \leqslant p < \infty$, with $\mathbb{V}\tilde{Y}$ a section of $\mathcal{L}_2(\mathcal{H}; \mathcal{L}_2(\mathcal{H}; L^2([0,T]; \mathbf{R}^m)))$. Then

$$\mathbb{V}_{V^1} V^2 - \tilde{X}\bar{d}(\tilde{Y} V^2) V^1 = -\tilde{X}(\mathbb{V}_{V^1}\tilde{Y}) V^2$$

and so

$$\mathbb{T}(V^1, V^2) = \tilde{X}\big((\mathbb{V}_{V^2}\tilde{Y}) V^1 - (\mathbb{V}_{V^1}\tilde{Y}) V^2\big).$$

From this we see we can consider the torsion as a section of $\mathcal{L}_2(\bigwedge^2 \mathcal{H}; TC_{x_0}M)$. Alternatively noting that $\tilde{Y}$ maps $\mathcal{H}$ into $C_0([0,T]; \mathbf{R}^m)$ and arguing as before we see that it gives a section of $\mathcal{L}_{skew}(\mathcal{H}, \mathcal{H}; TC_{x_0}M)$. In both cases the sections are in L^p for all $1 \leqslant p < \infty$.

Finally we give the result used in Section 8.

Proposition B.3. *If ϕ is a smooth cylindrical 1-form and V^1, V^2 are in $\mathbb{W}^{p,1}\mathcal{H}$ then, almost surely,*

$$2\, d\phi(V^1 \wedge V^2) = \iota_{V^1}\bar{d}\iota_{V^2}\phi - \iota_{V^2}\bar{d}\iota_{V^1}\phi - \phi([V^1, V^2]).$$

Proof. Using the pointwise connection on the sections of $T^*C_{x_0}M$:

$$\begin{aligned}&\iota_{V^1}\bar{d}\iota_{V^2}\phi - \iota_{V^2}\bar{d}\iota_{V^1}\phi - \phi([V^1, V^2]) \\ &\quad = (\tilde{\nabla}_{V^1}\phi)(V^2) + \phi(\tilde{\nabla}_{V^1}) V^2 - (\tilde{\nabla}_{V^2}\phi)(V^1) - \phi(\tilde{\nabla}_{V^2} V^1) - \phi([V^1, V^2]) \\ &\quad = \tilde{\nabla}_{V^1}\phi(V^2) - \tilde{\nabla}_{V^2}\phi(V^1) \\ &\quad = \frac{1}{2} d\phi(V^1, V^2)\end{aligned}$$

by the standard formula, as the pointwise connection $\tilde{\nabla}$ has no torsion. □

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